# A Complete Quadrilateral in Rectangular Coordinates

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**ABSTRACT**

A complete quadrilateral in the Euclidean plane is studied. The geometry of such quadrilateral is almost as rich as the geometry of a triangle, so there are lot of associated points, lines and conics. Hereby, the study was performed in the rectangular coordinates, symmetrically on all four sides of the quadrilateral with four parameters $a,b,c,d$. In this paper we will study the properties of some points, lines and circles associated to the quadrilateral. All these properties are well known, but here they are all proved by the same method. During this process, still some new results have appeared.

**Key words:** Euclidean plane, complete quadrilateral, parabola

**MSC2020:** 51N20

**Potpuni četverostran u pravokutnim koordinatama**

**SAŽETAK**

U radu proučavamo potpuni četverostran u euklidskoj ravnini. Poput trokuta i potpuni četverostran ima mnogo zanimljivih svojstava te pridruženih točaka, pravaca i konika. Ovdje je proučavanje provedeno korištenjem pravokutnih koordinata, simetrično po sve četiri stranice četverostrana s četiri parametra $a,b,c,d$. Proučavamo svojstva točaka, pravaca i kružnica pridruženih četverostranu. Gotovo sve tvrdnje prikazane u ovom radu su dobro poznate, ali su se ipak ponegdje usput pojavili i neki novi rezultati.

**Ključne riječi:** euklidska ravnina, potpuni četverostran, parabola

## 1 Motivation

The focus of this paper is the geometry of a complete quadrilateral in the Euclidean plane. Such a geometry is almost as rich as the geometry of a triangle, so there are lot of associated points, lines and conics. The facts given in the paper are well known, but the idea of the paper is to prove them all by the same method. Hence, the study is performed in the rectangular coordinates, symmetrically on all four sides of the quadrilateral with four parameters $a,b,c,d$. During this process, still some new results have appeared.

We mention only the literature where the facts and the statements are presented for the first time.

Previously known statements are included in the text and given in italic while the new results are given in the form of theorem.

## 2 Introduction

A complete quadrilateral, or just a *quadrilateral* $\mathcal{ABCD}$ is a set of four lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ in the Euclidean plane, where none of two lines are parallel and no three of which are concurrent. Lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are *sides* of that quadrilateral, and intersections of the pairs of lines are its *vertices*. Pairs of vertices $T_{AB} = \mathcal{A} \cap \mathcal{B}$, $T_{CD} = \mathcal{C} \cap \mathcal{D}$; $T_{AC} = \mathcal{A} \cap \mathcal{C}$, $T_{BD} = \mathcal{B} \cap \mathcal{D}$; $T_{AD} = \mathcal{A} \cap \mathcal{D}$, $T_{BC} = \mathcal{B} \cap \mathcal{C}$ are pairs of *opposite vertices*, and their connecting lines $\mathcal{U} = T_{AB}T_{CD}$, $\mathcal{V} = T_{AC}T_{BD}$, $\mathcal{W} = T_{AD}T_{BC}$ are *diagonals* of that quadrilateral. Intersection points $U = \mathcal{V} \cap \mathcal{W}$, $V = \mathcal{W} \cap \mathcal{U}$, $W = \mathcal{U} \cap \mathcal{V}$ are *diagonal points* and a triangle formed by diagonal points and diagonals is a *diagonal triangle* of a quadrilateral. Only one parabola $\mathcal{P}$ can be inscribed to the quadrilateral $\mathcal{ABCD}$ and let it touches the sides of the

quadrilateral at the points $A$, $B$, $C$, $D$. An axis and a vertex tangent of that parabola is taken as $x$-axis and $y$-axis of the coordinate system. Then, taking any metrical unit for length the equation of that parabola is $y^2 = 2px$. That parabola has the point $(\frac{p}{2}, 0)$ as a focus and the line $x = -\frac{p}{2}$ as the directrix. Without loss of generality, we can take the metrical unit for length in a way that $p = 2$ is valid. The size of some object is not important, but only its shape and mutually position to the similarity. The diagonal triangle is autopolar with respect to the parabola $\mathcal{P}$, i.e. lines $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ are polars of points $U, V, W$ with respect to parabola.
Hence, we can take that inscribed parabola $\mathcal{P}$ of the quadrilateral $\mathcal{ABCD}$ has the equation

$$\mathcal{P} \ldots y^2 = 4x, \tag{1}$$

so its focus is point $S = (1, 0)$, the directrix $\mathcal{H}$ is $x = -1$. The polarity with respect to the parabola $\mathcal{P}$ maps any point $T_0 = (x_0, y_0)$ to the line $\mathcal{T}_0$ with the equation $y_0 y = 2x + 2x_0$, the polar line of the point $T_0$. For the contact points of the parabola $\mathcal{P}$ with the sides of the quadrilateral the following points are taken

$$A = (a^2, 2a), B = (b^2, 2b), C = (c^2, 2c), D = (d^2, 2d). \tag{2}$$

The tangent lines of the parabola at the points $A, B, C, D$ are lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ with equations

$$\begin{aligned} &\mathcal{A} \ldots ay = x + a^2, \quad \mathcal{B} \ldots by = x + b^2, \\ &\mathcal{C} \ldots cy = x + c^2, \quad \mathcal{D} \ldots dy = x + d^2, \end{aligned} \tag{3}$$

because on the example of the polar line for the point $A$ we get the equation $2ay = 2x + 2a^2$. For the vertices of the quadrilateral $\mathcal{ABCD}$ we get the following forms

$$\begin{aligned} &T_{AB} = (ab, a+b), T_{AC} = (ac, a+c), T_{AD} = (ad, a+d), \\ &T_{CD} = (cd, c+d), T_{BD} = (bd, b+d), T_{BC} = (bc, b+c). \end{aligned} \tag{4}$$

The diagonals are

$$\begin{aligned} \mathcal{U} &= T_{AB}T_{CD} \ldots (cd - ab)y \\ &= (c + d - a - b)x + (a + b)cd - ab(c + d), \\ \mathcal{V} &= T_{AC}T_{BD} \ldots (bd - ac)y \\ &= (b + d - a - c)x + (a + c)bd - ac(b + d), \\ \mathcal{W} &= T_{AD}T_{BC} \ldots (bc - ad)y \\ &= (b + c - a - d)x + (a + d)bc - ad(b + c), \end{aligned} \tag{5}$$

and diagonal points

$$\begin{aligned} U &= \left( \frac{(a+b)cd - ab(c+d)}{c+d-a-b}, 2\frac{cd - ab}{c+d-a-b} \right), \\ V &= \left( \frac{(a+c)bd - ac(b+d)}{b+d-a-c}, 2\frac{bd - ac}{b+d-a-c} \right), \\ W &= \left( \frac{(a+d)bc - ad(b+c)}{b+c-a-d}, 2\frac{bc - ad}{b+c-a-d} \right). \end{aligned} \tag{6}$$

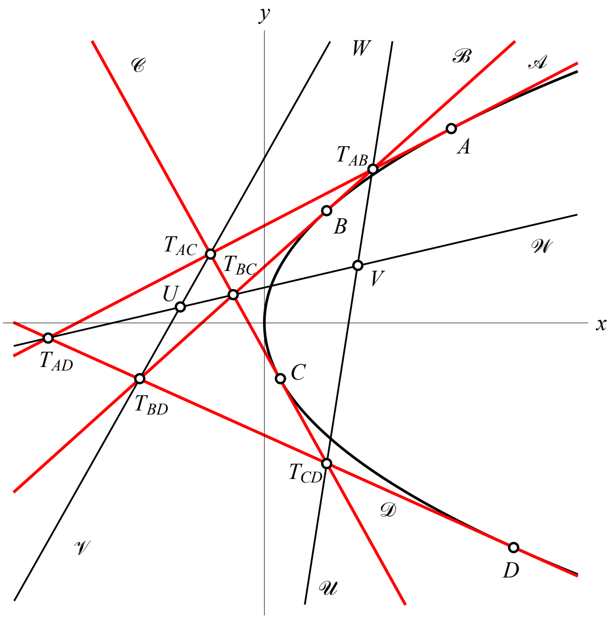


Figure 1: *A complete quadrilateral $\mathcal{ABCD}$*

There is a complete quadrilateral $\mathcal{ABCD}$ on Figure 1.
Let denote basic symmetric functions of the parameters $a, b, c, d$ by $s, q, r$ and $p$, so that

$$\begin{aligned} &s = a + b + c + d, \quad q = ab + ac + ad + bc + bd + cd, \\ &r = abc + abd + acd + bcd, \quad p = abcd \end{aligned}$$

are valid.

We will often use and labels $\alpha = a^2 + 1$, $\beta = b^2 + 1$, $\gamma = c^2 + 1$, $\delta = d^2 + 1$.

## 3 On a complete quadrilateral

Hereby, we will give many well known results on a complete quadrilateral $\mathcal{ABCD}$, as well as few new results. Connecting line $AB$ from (2) is the line with the equation $2x - (a + b)y + 2ab = 0$ that is fullfilled by coordinates of the point $U$ from (6). Similarly computation is valid for others connecting lines of the points $A$, $B$, $C$ and $D$. Hence, *a complete quadrangle ABCD has the same diagonal triangle UVW as the quadrilateral $\mathcal{ABCD}$*, see [43].
The midpoints of pairs of points $T_{AB}, T_{CD}$; $T_{AC}, T_{BD}$; $T_{AD}, T_{BC}$ from (4) are following points

$$\begin{aligned} &U_0 = \left( \frac{1}{2}(ab + cd), \frac{1}{2}s \right), \quad V_0 = \left( \frac{1}{2}(ac + bd), \frac{1}{2}s \right), \\ &W_0 = \left( \frac{1}{2}(ad + bc), \frac{1}{2}s \right) \end{aligned} \tag{7}$$

and obviously they lie on the line $\mathcal{N}$ with the equation

$$\mathcal{N} \ldots y = \frac{1}{2}s. \tag{8}$$

There are lots of names for this line, here we will call it a *median* of the quadrilateral $\mathcal{ABCD}$. It was mentioned for the first time in [9], an its existence was proved in [16]. Out of formulas (7) the following formulas for directed lengths follow

$$V_0W_0 = \frac{1}{2}(a-b)(d-c), \quad W_0U_0 = \frac{1}{2}(a-c)(b-d),$$
$$U_0V_0 = \frac{1}{2}(a-d)(c-b). \tag{9}$$

The centroid of six points from (4) and the centroid of four points from (2) are points

$$T = (\frac{1}{6}q, \frac{1}{2}s), \quad T' = (\frac{1}{4}(s^2-2q), \frac{1}{2}s) \tag{10}$$

that are incident with $\mathcal{N}$.

In [33] the following statement is proved:
*Areas of two triangles whose bases are two diagonals of the quadrilateral, and common vertex is any of two additional vertices of that quadrilateral, are related as segments on the median of the quadrilateral from the midpoints of these two vertices to the midpoints of two diagonals mentioned before.*
That means that areas of triangles $T_{AB}T_{AC}T_{BD}$ and $T_{AB}T_{AD}T_{BC}$, as well as areas of triangles $T_{CD}T_{AC}T_{BD}$ and $T_{CD}T_{AD}T_{BC}$ are related as directed lengths $U_0V_0$ and $U_0W_0$, and there are two more such examples. Areas of triangles $T_{AB}T_{AC}T_{BD}$ and $T_{AB}T_{AD}T_{BC}$ are $\frac{1}{2}(a-b)(a-d)(b-c)$ and $\frac{1}{2}(a-b)(a-c)(b-d)$, and areas of $T_{CD}T_{AC}T_{BD}$ and $T_{CD}T_{AD}T_{BC}$ are $\frac{1}{2}(c-d)(a-d)(b-c)$ and $\frac{1}{2}(c-d)(a-c)(b-d)$ while directed lengths $U_0V_0$ and $U_0W_0$ according to (9) are equal $\frac{1}{2}(a-d)(c-b)$ and $\frac{1}{2}(a-c)(d-b)$. So, all three mentioned ratios are equal to $\frac{(a-d)(b-c)}{(a-c)(b-d)}$, that is actually the cross ratio $(abdc)$. In another two examples two cross ratios are $(acbd)$ and $(adcb)$.
In the quadrilateral $\mathcal{ABCD}$ we can observe three quadrangles $T_{AC}T_{AD}T_{BD}T_{BC}$, $T_{AB}T_{AD}T_{CD}T_{BC}$, and $T_{AB}T_{AC}T_{CD}T_{BD}$, one of them is convex, one concave, and one crossed. Centroids of these quadrangles are points

$$T_u = \left(\frac{1}{4}(ac+ad+bc+bd), \frac{1}{2}s\right),$$
$$T_v = \left(\frac{1}{4}(ab+ad+bc+cd), \frac{1}{2}s\right),$$
$$T_w = \left(\frac{1}{4}(ab+ac+bd+cd), \frac{1}{2}s\right)$$

incident with the median $\mathcal{N}$. For the directed lengths on that line we have the equalities

$$T_vT_w = \frac{1}{4}(a-b)(c-d),$$
$$T_wT_u = \frac{1}{4}(a-c)(d-b),$$
$$T_uT_v = \frac{1}{4}(a-d)(b-c),$$

so because of the equality (9) we get $V_0W_0 = -2T_vT_w$, $W_0U_0 = -2T_wT_u$, $U_0V_0 = -2T_uT_v$ that is a result given in [32].
If for oriented lengths equalities $T_{AB}T_{AD} = uT_{AB}T_{AC}$, $T_{AB}T_{BC} = vT_{AB}T_{BD}$ are valid, then easily we get equalities $\frac{b-d}{b-c} = u$, $\frac{a-c}{a-d} = v$. If the number $w$ is such that $U_0W_0 = wU_0V_0$ is fulfilled, then because of (9)

$$w = \frac{(a-c)(d-b)}{(a-d)(c-b)} = uv$$

follows. This is result from [38].
Let the points $B_1$ and $C_1$ be points on lines $\mathcal{B}$ and $\mathcal{C}$ so that for the directed lengths the equalities $T_{AB}B_1 = T_{BC}T_{BD}$, $T_{AC}C_1 = T_{BC}T_{CD}$ are valid. Then out of (4) we get points $B_1$ and $C_1$ of the form

$$B_1 = (ab+bd-bc, a+b-c+d),$$
$$C_1 = (ac+cd-bc, a-b+c+d).$$

The line $B_1C_1$ has the equation $2x-(a+d)y+(a+d)^2-(a+d)(b+c)+2bc = 0$, and its intersections $A_1$ and $D_1$ with lines $\mathcal{A}$ and $\mathcal{D}$ with equations (3) are points having ordinates

$$\frac{1}{d-a}[d^2+2ad-a^2-(a+d)(b+c)+2bc],$$
$$\frac{1}{a-d}[a^2+2ad-d^2-(a+d)(b+c)+2bc],$$

so the midpoint of these points $A_1$ and $D_1$ has an ordinate $a+d$, the midpoint of $B_1$ and $C_1$ has the same ordinate as well. Because of that *directed lengths $A_1B_1$ and $C_1D_1$ are equal*. This statement we find in [35]. We see that: *the common midpoint of the line segments $A_1D_1$ and $B_1C_1$ is incident with the diameter of the parabola $\mathcal{P}$ through the point $T_{AD}$*. There are five more analogous statements where we find common midpoints of the pairs of line segments on diameters of parabola $\mathcal{P}$ through other five vertices of the quadrilateral $\mathcal{ABCD}$.
The median $\mathcal{N}$ with the equation $y = \frac{1}{2}(a+b+c+d)$ intersects the line $\mathcal{A}$ with the equation $ay = x+a^2$ in the point $\mathcal{A}\cap\mathcal{N} = (\frac{1}{2}(ab+ac+ad-a^2), \frac{1}{2}(a+b+c+d))$, and a midpoint of that point and point $T_{BC} = (bc, b+c)$ is the point

$$\left(\frac{1}{4}(ab+ac+ad-a^2+2bc), \frac{1}{4}(a+3b+3c+d)\right).$$

This midpoint is incident with the line

$$2x-(a+b+c-d)y+\frac{1}{4}(3a^2+3b^2+3c^2-d^2+2ab+$$
$$+2ac+2bc-2ad-2bd-2cd)=0.$$

Midpoints of the pairs $\mathcal{B}\cap\mathcal{N}, T_{AC}$ and $\mathcal{C}\cap\mathcal{N}, T_{AB}$ are incident with it as well. Hence, that line is a median of the quadrilateral $\mathcal{ABCN}$. Its intersection with the line $\mathcal{N}$ is the point $(\frac{1}{2}(4q-s^2), \frac{1}{2}s)$ that is incident with medians of quadrilaterals $\mathcal{ABDN}$, $\mathcal{ACDN}$, $\mathcal{BCDN}$. It is point $QL-P23$ in [43].

Points symmetric to intersections of the given line with sides of given triangle with respect to the midpoints of these lines are incident with one line that are said to be *reciprocal* to given line with respect to given triangle.

Let us determine the line $\mathcal{D}'$ reciprocal to the line $\mathcal{D}$ with respect to trilateral $\mathcal{ABC}$. A point on the line $\mathcal{A}$ symmetric to the point $T_{AD}$ with respect to the midpoints $T_{AB}$ and $T_{AC}$ is of the form $(ab+ac-ad, a+b+c-d)$. Out of symmetry on $a,b,c$ of the ordinate of that point we conclude that the same ordinate is achieved in a similar procedure with lines $\mathcal{B}$ and $\mathcal{C}$. So, *the equation of $\mathcal{D}'$ is $y=a+b+c-d$ and it is parallel to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$.* This statement is coming from both [17] and [31]. Analogously, lines $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ reciprocal to lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ with respect to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ have equations $y=-a+b+c+d$, $y=a-b+c+d$, $y=a+b-c+d$. Adding up these four equations, we find $4y=2(a+b+c+d)$, i.e. the equation $y=\frac{1}{2}s$ of the median $\mathcal{N}$. Hence, the median of the quadrilateral $\mathcal{ABCD}$ is so-called *centroid line* of the lines $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$.

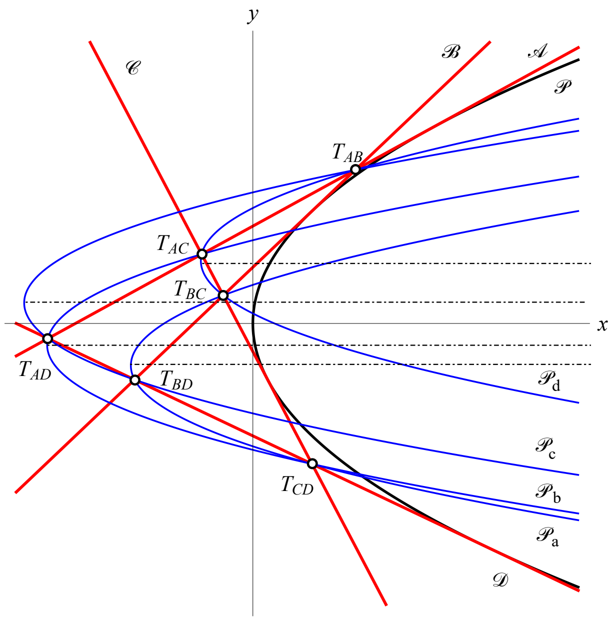


Figure 2: *Parabolas circumscribed to trilaterals*

*Lines $\mathcal{D}$ and $\mathcal{D}'$ are reciprocal with respect to trilateral $\mathcal{ABC}$* and that relationship is symmetric on that two lines, so as *the line $\mathcal{D}'$ is parallel to the median of the quadrilateral $\mathcal{ABCD}$, then and the line $\mathcal{D}$ is parallel to the median of the quadrilateral $\mathcal{ABCD}'$.* Similar is valid for the other sides of the quadrilateral $\mathcal{ABCD}$ and their reciprocal lines $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$. This statement is found in [39].

In [12] it is stated:

*Parabolas inscribed to trilaterals formed by three sides of the quadrilateral with axes parallel to the axis of parabola inscribed to that quadrilateral arises from this inscribed parabola by using translations.*

The statement is not quite correct.

Namely, parabola $\mathcal{P}_d$ with equation

$$y^2-(a+b+c)y=x-ab-ac-bc$$

is incident with points $T_{AB}, T_{AC}, T_{BC}$, i. e. it is circumscribed to the trilateral $\mathcal{ABC}$, and it has an axis parallel to the axis of $\mathcal{P}$, but its parameter is equal to the quarter of the parameter of $\mathcal{P}$. See Figure 2. Hence the following theorem is valid:

**Theorem 1** *The parabolas $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$, $\mathcal{P}_d$ inscribed to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ arise from each other by translations. The parameter of these parabolas is equal to the quarter of the parameter of $\mathcal{P}$ .*

For example, substituting

$$x\to x+\frac{1}{4}(c-d)(2a+2b-c-d),$$
$$y\to y+\frac{1}{2}(c-d)$$

the equation of $\mathcal{P}_d$ turns into $y^2-(a+b+d)y=x-ab-ad-bd$ that is the equation of $\mathcal{P}_c$.

A parabola with an equation $12x=9y^2-6sy+4q$ passes through the centroid $G_d=(\frac{1}{3}(ab+ac+bc), \frac{2}{3}(a+b+c))$ of the trilateral $\mathcal{ABC}$, and then through centroids of other three trilaterals of the quadrilateral $\mathcal{ABCD}$. A vertex of this parabola is the point $(\frac{1}{12}(4q-s^2), \frac{1}{3}s)$, and its axis has the equation $y=\frac{1}{3}s$, so *the distance from the focus $S$ of the quadrilateral $\mathcal{ABCD}$ to its median is equal to three halves of the distance from this focus to this parabola.* This statement is from [5] and [6].

The parabola $\mathcal{P}$ inscribed to the quadrilateral $\mathcal{ABCD}$ is circumscribed to the quadrangle $ABCD$. However, there is one more parabola circumscribed to that quadrangle. It is parabola with the equation

$$x^2-\frac{1}{2}sxy+\frac{1}{16}s^2y^2+(q-\frac{1}{4}s^2)x-\frac{1}{2}ry+p=0,$$

because for example for the point $A=(a^2,2a)$ we get equality $a^4-a^3s+a^2q-ar+p=0$. The square part of previous equation is $\frac{1}{16}(4x-sy)^2$, so it follows that the axis of this parabola has the slope $\frac{4}{s}$. The slope of connecting line of the focus $S=(1,0)$ and the intersection point $Q=(-1,\frac{1}{2}s)$ of the median and the directrix is equal to $\frac{-s}{4}$. It proves that *the axis of studied parabola is perpendicular to the connecting line $SQ$*, see [43].

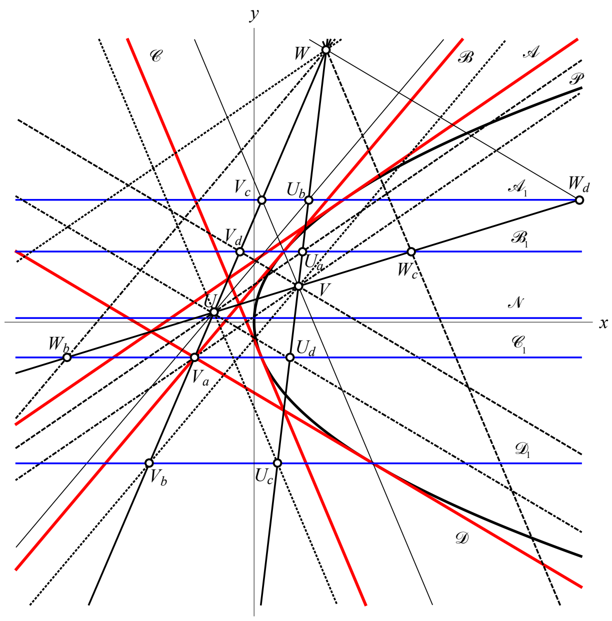


Figure 3: *Lines $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{D}_1$ are parallel to the median $\mathcal{N}$*

A line through the point $U$ from (6), parallel to the line $\mathcal{B}$ has the equation

$$(c+d-a-b)(x-by)=2ab^2+acd-abc-abd-bcd$$

and intersects the line $\mathcal{U}$ from (5) in the point $\mathcal{U}_b$ with coordinates

$$x=\frac{1}{c+d-a-b}(abc+abd+acd-bcd-2a^2b),$$
$$y=2a.$$

Analogously, a line parallel to the line $\mathcal{C}$ and incident with $V$ intersects a line $\mathcal{V}$ in the point $V_c$ with coordinates

$$x=\frac{1}{b+d-a-c}(abc+abd+acd-bcd-2a^2c),$$
$$y=2a,$$

and the line parallel to the line $\mathcal{D}$ and incident with $W$ intersects a line $\mathcal{W}$ in the point $W_d$ with coordinates

$$x=\frac{1}{b+c-a-d}(abc+abd+acd-bcd-2a^2d),$$
$$y=2a.$$

*The points $U_b,V_c,W_d$ are incident with one line $\mathcal{A}_1$ with the equation $y=2a$. Analogously, sets of three similar points $U_a,V_d,W_c$; $U_d,V_a,W_b$; $U_c,V_b,W_a$ are incident with lines $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{D}_1$, respectively, and lines $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{D}_1$ are parallel (see Figure 3).* This is statement from [26]. During this process, the new result has appeared:

**Theorem 2** *All four lines $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{D}_1$ are parallel to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$, and the median is their centroid line.*

Altitudes from the vertices $T_{BC},T_{AC},T_{AB}$ in the triangle $T_{BC}T_{AC}T_{AB}$ have equations $y=-ax+b+c+abc$, $y=-bx+a+c+abc$, $y=-cx+a+b+abc$ and they are intersected in the point $H_d=(-1,a+b+c+abc)$ that is orthocenter of that triangle, i. e. of the trilateral $\mathcal{ABC}$. Similarly, orthocenters of trilaterals $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$ are points

$$H_c=(-1,a+b+d+abd),$$
$$H_b=(-1,a+c+d+acd),$$
$$H_a=(-1,b+c+d+bcd).$$

All four orthocenters lie on the line $\mathcal{H}$ with the equation

$$\mathcal{H}\ldots x=-1, \tag{11}$$

which is the directrix of the inscribed parabola $\mathcal{P}$ of the quadrilateral $\mathcal{ABCD}$ and they have the centroid $G_h=(-1,\frac{1}{4}(3s+r))$. The statement that *these four orthocenters are incident with one line* is given in [37] without proof. The proof is given in [23]. In the literature, the line $\mathcal{H}$ has many names, herein we will call it a *directrix* of the quadrilateral $\mathcal{ABCD}$. The intersection point of the median and the directrix of the quadrilateral $\mathcal{ABCD}$ is the point

$$Q=(-1,\frac{1}{2}s) \tag{12}$$

which is called $QL-P7$ Newton-Steiner point in [43]. The midpoint of this point and the focus $S=(1,0)$ is the point $(0,\frac{1}{4}s)$ that is in [43] denoted by $QL-P19$.
The line through $H_a=(-1,b+c+d+bcd)$, parallel to the line $\mathcal{A}$ has the equation $x-ay+1+ab+ac+ad+abcd=0$ and goes through the point $(-1-ab-abcd,c+d)$ where the line $\mathcal{H}_b$, parallel to $\mathcal{B}$ passes as well. The line through $H_a$, perpendicular to $\mathcal{A}$ has the equation $ax+y=-a+b+c+d+bcd$ and goes through the point $(-2-cd,a+b+c+d+acd+bcd)$, through which the line perpendicular to $\mathcal{B}$ through the point $H_b$ passes as well. The connecting line of two obtained points has the equation

$$(a+b)x+(1-ab)y+a+b-c-d+a^2b+ab^2+$$
$$+abc+abd+a^2bcd+ab^2cd=0$$

and passes through the point $(-2-abcd,a+b+c+d)$. Five analogous lines are incident with that point as well.

Hence, *quadrilaterals, that are formed by the lines that passes through points $H_a$, $H_b$, $H_c$, $H_d$, parallel to lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, and perpendicular to these lines, are perspective.* A centre of the perspectivity is the point $(-p-2,s)$, which in [43] is called $QL-P21$ adjunct orthocenter homothetic center, although this is not homothecy. The point $(-1-ab-abcd,c+d)$ and the point $T_{AB}=(ab,a+b)$ have for the midpoint the point $(\frac{1}{2}(1+abcd),\frac{1}{2}(a+b+c+d))$, and similar is valid for five more pairs of corresponding points. It means that *the quadrilateral, formed by parallels to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ through the points $H_a$, $H_b$, $H_c$, $H_d$ is symmetric to the quadrilateral $\mathcal{ABCD}$ with respect to the point $(-\frac{1}{2}(1+p),\frac{1}{2}s)$ that is in [43] called $QL-P20$ orthocenter homothetic center. It is obviously incident with the median $\mathcal{N}$.*

In the quadrilateral $T_{AB}T_{BC}T_{BD}T_{AD}$ perpendiculars from points $T_{AC}$ and $T_{BC}$ to the line $\mathcal{D}$ pass through orthocenters $H_a$ and $H_b$ of trilaterals $\mathcal{BCD}$ and $\mathcal{ACD}$, hence the perpendicular from the midpoint of the side $T_{AC}T_{BC}$ to the opposite side $T_{BD}T_{AD}$ passes through the midpoint of $H_a$ and $H_b$. In the same way, the perpendiculars from $T_{BD}$ and $T_{AD}$ to the line $\mathcal{D}$ pass through the orthocenters $H_a$ and $H_b$ and because of that the perpendicular from the midpoint of $T_{BD}T_{AD}$ to the opposite side $T_{AC}T_{BC}$ passes through the midpoint of $H_a$ and $H_b$. So, *for the pair of opposite sides $T_{AC}T_{BC}$, $T_{BD}T_{AD}$ perpendiculars from the midpoint of the each of them to the opposite side are intersected in one point on the directrix, which is the midpoint of $H_a$ and $H_b$.* In the same way it is shown that for *the pair of opposite sides $T_{AC}T_{AD}$, $T_{BD}T_{BC}$ perpendiculars from the midpoint of the each of them to the opposite side are intersected in one point on the directrix, which is the midpoint of $H_c$ and $H_d$.* Analogously it is valid for the quadrangles $T_{AB}T_{AD}T_{CD}T_{BC}$ and $T_{AB}T_{AC}T_{CD}T_{BD}$, so we get four more times per two lines, that are intersected in midpoints of pairs of orthocenters $H_a,H_c$ and $H_b,H_d$, and $H_a,H_d$ and $H_b,H_c$.
The distance of the focus $S$ of the quadrilateral $\mathcal{ABCD}$ to its median $\mathcal{N}$ and to its directrix $\mathcal{H}$ are equal to $\frac{1}{2}s$ and 1, so their ratio is $\frac{1}{2}s=\frac{1}{2}(a+b+c+d)$. However, for example the line $\mathcal{A}$ has an equality $\cot\angle(\mathcal{N},\mathcal{A})=a$, so that *the ratio we have mentioned is equal to* $\frac{1}{2}[\cot\angle(\mathcal{N},\mathcal{A})+\cot\angle(\mathcal{N},\mathcal{B})+\cot\angle(\mathcal{N},\mathcal{C})+\cot\angle(\mathcal{N},\mathcal{D})]$ that is statement found in [27].

If $\mathcal{L}$ is line having equation $y=mx+n$, then the perpendicular from $T_{AB}$ to that line has the equation $x+my=ab+am+bm$, and the intersection point of that two lines has the coordinates $x=\frac{1}{\eta}(am+bm-mn+ab), y=\frac{1}{\eta}(am^2+bm^2+abm+n)$, where $\eta=m^2+1$. The perpendicular from that intersection point to line $\mathcal{C}$ has the equation $\eta(cx+y)=acm+bcm-cmn+abc+am^2+bm^2+abm+n$. It can be checked that this line passes through the point with coordinates

$$x=-\frac{1}{\eta}(m^2+mn), \tag{13}$$
$$y=\frac{1}{\eta}[(a+b+c)m^2+(ab+ac+bc)m+abc+n].$$

The perpendiculars to lines $\mathcal{A}$ and $\mathcal{B}$ from pedals of perpendiculars from points $T_{BC}$ and $T_{AC}$ to the line $\mathcal{L}$ are incident with the point (13).
Because of that this point is an orthopole of the line $\mathcal{L}$ with respect to the trilateral $\mathcal{ABC}$. Similarly, the same is valid for the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$. Hence, *all four orthopoles are incident with the line having equation $x=-\frac{1}{\eta}(m^2+mn)$ that is perpendicular to the median of the quadrilateral $\mathcal{ABCD}$.* This statement is from [21]. *If for the line $\mathcal{L}$ the line $\mathcal{D}$ is taken, then $m=\frac{1}{d}, n=d, \eta=\frac{1}{\delta}\cdot d^2$ are valid, so for the orthopole of the line $\mathcal{D}$ with respect to the trilateral $\mathcal{ABC}$ we get the point*

$$O_d=\Big(-1,\frac{1}{\delta}[a+b+c+(ab+ac+bc)d+abcd^2+d^3]\Big), \tag{14}$$

*that is incident with the directrix of the quadrilateral $\mathcal{ABCD}$ as well as the analogous orthopoles of lines $\mathcal{A},\mathcal{B},\mathcal{C}$ with respect to the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$.* This statement is coming from [18]. The same statement can be found in [28], but herein the author observes on these orthopoles as radical centers of pedal circles on the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ with respect to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, what is in accordance with so-called Lemoyne's theorem (see [18]).
In the previous proof it was assumed that $m\neq 0$. Let the line $\mathcal{L}$ be parallel with the median and with the equation $y=n$. The pedal point of the perpendicular from the point $T_{AB}$ to that line is the point $(ab,n)$, and perpendicular from that point to the line $\mathcal{C}$ has the equation $cx+y=abc+n$. This perpendicular, and two more analogous perpendiculars, are incident with the point $(0,abc+n)$ that is an orthopole of $\mathcal{L}$ with respect to the trilateral $\mathcal{ABC}$. This orthopole and orthoploes of the line $\mathcal{L}$ with respect to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ are incident with the $y$-axis, the vertex tangent of the parabola $\mathcal{P}$.
Let us study any line $\mathcal{L}$ parallel to the directrix with the equation $x=l$. The pedal point of the perpendicular from $T_{AB}$ to that line is the point $(l,a+b)$, and the perpendicular from that point to the line $\mathcal{C}$ has the equation $cx+y=cl+a+b$ and obviously it passes through the point $(l-1,a+b+c)$, that is orthopole of the line $\mathcal{L}$ with respect to the trilateral $\mathcal{ABC}$. As well as other three orthopoles of $\mathcal{L}$ with respect to the trilateral $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$, it is incident to the line with the equation $x=l-1$, parallel to the median of $\mathcal{ABCD}$ and the line $\mathcal{L}$. Particularly, *there*

*are orthopoles of the vertex tangent of the parabola $\mathcal{P}$ with respect to trilateras $\mathcal{ABC}$, $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$ on the directrix*. These statements are in [43] but they are atributed to S. Kirikami.

The line $\mathcal{L}'$, where orthopoles of the given line $\mathcal{L}$ with respect to the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ are lying, is called an *orthopolar line* of the line $\mathcal{L}$ with respect to the quadrilateral $\mathcal{ABCD}$.

Earlier, we found out that the line $\mathcal{L}$ with the equation $y = mx+n$ has the orthopolar $\mathcal{L}'$ with the equation $x = -\frac{1}{\eta}(m^2+mn)$, that line $\mathcal{L}$ with the equation $y = n$ has the orthopolar $\mathcal{L}'$ with the equation $x = 0$, and the line $\mathcal{L}$ with the equation $x = l$ has the orthopolar $\mathcal{L}'$ with the equation $x = l-1$. The tangent at the point $(m^2,-2m)$ at parabola $\mathcal{P}$ has the equation $x+my+m^2 = 0$ and it is perpendicular to the given line $\mathcal{L}$ with the equation $y = mx+n$. These two lines has the intersection with the abscissa $x = -\frac{1}{\eta}(m^2+mn)$, that lies on the line $\mathcal{L}'$. The tangent line of the parabola $\mathcal{P}$ at the point $(t^2,2t)$ has the equation $ty = x+t^2$, i.e. $m = \frac{1}{t}$, $n = t$. Because of that the orthopolar of that tangent has the equation $x = -1$, and that is directrix $\mathcal{H}$. So, *the orthopolar of any tangent of the parabola $\mathcal{P}$ is the directrix $\mathcal{H}$*. If the line $\mathcal{L}$ passes through the focus $S$, then it has the equation $y = m(x-1)$, and for it $n = -m$ is valid and orthopolar $\mathcal{L}'$ has the equation $x = 0$ and that is vertex tangent $\mathcal{Y}$ of the parabola $\mathcal{P}$.

Let $\mathcal{P}''$ be parabola, with the same focus $S = (1,0)$ and the same axis as parabola $\mathcal{P}$. If its directrix has the equation $x = t$, then that parabola $\mathcal{P}''$ has the equation $(x-1)^2+y^2 = (x-t)^2$, that after simplifying, reaches the form $y^2 = 2(1-t)x+t^2-1$. The intersections of this parabola and parabola $\mathcal{P}$ with equation $y^2 = 4x$ are the points $(x',y') = (\frac{1}{2}(t-1), \pm\sqrt{2(t-1)})$, where $t > 1$. Tangents at these points to both parabolas have the slopes $\frac{2}{y'}$ and $\frac{1-t}{y'}$, whose product is equal to $-1$, because $y'^2 = 2(t-1)$. That is the reason why those two parabolas are orthogonal which is special case of very well known fact that confocal conics are orthogonal. Let $(x'',y'')$ be any point of parabola $\mathcal{P}''$. Tangent at this point to this parabola has the equation $yy' = (1-t)(x+x'')+t^2-1$, so because of it $m = \frac{1-t}{y''}, n = 1-ty''(x''-t-1)$ are valid and we get

$$\begin{aligned} y''^2(m^2+mn) &= (1-t)^2(x''-t), \\ y''^2(m^2+1) &= (1-t)^2+y''^2 \\ &= (1-t)^2+2(1-t)x''+t^2-1 \\ &= 2(1-t)(x''-t), \end{aligned}$$

where we use the equality $y''^2 = 2(1-t)x''+t^2-1$, because the point $(x'',y'')$ is incident with parabola $\mathcal{P}''$. Because of this $-\frac{1}{\eta}(m^2+mn) = \frac{1}{2}(t-1)$ is valid, so each tangent line of parabola $\mathcal{P}$ has the same orthopolar with the equation $x = \frac{1}{2}(t-1)$ that passes through the intersections $(\frac{1}{2}(t-1), \pm\sqrt{2(t-1)})$ of parabolas $\mathcal{P}$ and $\mathcal{P}''$. Hence, *all lines with the same orthopolar $\mathcal{L}$ perpendicular to median $\mathcal{N}$ are tangents to parabola $\mathcal{P}''$ which has the same focus and the same axis as parabola $\mathcal{P}$ and it is orthogonal to it at the intersection points with the line $\mathcal{L}'$*. These statements found in [43] are attributed to T. Q. Hung. The line $\mathcal{L}'$ with the equation $x = \frac{1}{2}(t-1)$ is the bisector of directrices of $\mathcal{P}$ and $\mathcal{P}'$ that have equations $x = -1$ and $x = t$.

The circle $\mathcal{S}_d$ through the points $T_{BC}, T_{AC}, T_{AB}$ from (4), i.e. the circumscribed circle to the trilateral $\mathcal{ABC}$, has the equation

$$\begin{aligned} &x^2+y^2-(ab+ac+bc+1)x-(a+b+c-abc)y+ \\ &+ab+ac+bc = 0, \end{aligned} \tag{15}$$

the center

$$S_d = \Big(\frac{1}{2}(ab+ac+bc+1), \frac{1}{2}(a+b+c-abc)\Big), \tag{16}$$

and the radius $\rho_d$ given by $4{\rho_d}^2 = (ab+ac+bc-1)^2+(a+b+c-abc)^2$, that is actually the formula

$$4{\rho_d}^2 = (a^2+1)(b^2+1)(c^2+1). \tag{17}$$

The circles $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c$ circumscribed to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, respectively, have the similar equations. The circle (15) passes through the point

$$S = (1,0), \tag{18}$$

that is focus of inscribed parabola $\mathcal{P}$ of the quadrilateral $\mathcal{ABCD}$, here we will call it a *focus* of this quadrilateral, although there are different names in the literature. W. Wallace has the fact that *four mentioned circles are incident with one point* in [34].

The point $P20 = (-\frac{1}{2}(p+1), \frac{1}{2}s)$ is the midpoint of the focus $S = (1,0)$ and the point $P21 = (-p-2,s)$, and the point $P19 = (0, \frac{1}{4}s)$ is the midpoint of the point $S$ and the point $Q$ from (12), because of that *lines $P20P19$ and $P21Q$ are parallel* (see [43]).

The line through the point $T_{BC}$ parallel to $\mathcal{A}$ has the equation $x-ay = bc-ab-ac$, and a connecting line $AS$ of the points $A = (a^2,2a)$ and $S = (1,0)$ has the equation $2ax+(1-a^2)y = 2a$. Those lines are intersected in

$$\begin{aligned} \Big(&\frac{1}{\alpha}(a^3b+a^3c-a^2bc+2a^2-ab-ac+bc), \\ &\frac{1}{\alpha}(2a-2abc+2a^2b+2a^2c)\Big) \end{aligned}$$

that lies on the circle $\mathcal{S}_d$ with the equation (15). Hence, *the parallel line to the line $\mathcal{A}$ through the point $T_{BC}$ intersects the circle $\mathcal{S}_d$ residually (except at the point $T_{BC}$) at the point on the line AS, and then by analogy, the other two intersection points of the circles $\mathcal{S}_b$ and $\mathcal{S}_c$ with lines through the points $T_{CD}$ and $T_{BD}$ parallel to the line $\mathcal{A}$ are incident with* *that line too. Similarly, we have three points each on lines BS, CS and DS.* This is statement from [7] and [8].

The line $\mathcal{A}$ with the equation $x-ay+a^2=0$ intersects the $y$-axis with the equation $x=0$ at the point $(0,a)$, and line through this point has generally the equation $mx-y+a=0$. The bisector of this line and the axis $y$-axis has the equation $\frac{1}{\sqrt{\eta}}(mx-y+a)\pm x=0$, where $\eta=m^2+1$, i.e. the equation $(m\pm\sqrt{\eta})x-y+a=0$ holds. This bisector is the same as the line $\mathcal{A}$ under the condition $a(m\pm\sqrt{m^2+1})=1$, out of which $m=\frac{1}{2a}(1-a^2)$. That's the reason why the line symmetric to the $y$-axis with respect to the line $\mathcal{A}$ has the equation $(a^2-1)x+2ay=2a^2$. The line symmetric to the $y$-axis with respect to the line $\mathcal{B}$ has the equation $(b^2-1)x+2by=2b^2$, and these two lines have the intersection $(\frac{2ab}{ab+1},\frac{a+b}{ab+1})$. This intersection point lies on the line with the equation $(a+b)x+(1-ab)y=a+b$, where the points $S=(1,0)$ and $T_{AB}=(ab,a+b)$ lie as well. Hence, *the line $ST_{AB}$ passes through the intersection of lines, that are symmetrical to the lines $\mathcal{A}$ and $\mathcal{B}$ with respect to the $y$-axis, a vertex tangent of parabola $\mathcal{P}$. Similar statement is valid for the lines $ST_{AC},ST_{AD},ST_{BC},ST_{BD},ST_{CD}$.* These statements are in [43] attributed to S. Kirikami.

*The perpendicular from the point $S=(0,1)$ to the line $\mathcal{A}$ has the equation $ax+y=a$, the parallel line $\mathcal{H}_{ab}$ with directrix $\mathcal{H}$ through the point $T_{CD}$ has the equation $x=cd$, and the intersection point of these lines is the point $(cd,a-acd)$, that lies on the circle $S_b$* with the equation $x^2+y^2-(ac+ad+cd+1)x-(a+c+d-acd)y+ac+ad+cd=0$, analogous to the one in (15). Similar to this, *the perpendicular from the point S to the line $\mathcal{B}$ intersects the line $\mathcal{H}_{ab}$ in the point $(cd,b-bcd)$ that lies on the circle $\mathcal{S}_b$. There are five more analogous lines $\mathcal{H}_{ac},\mathcal{H}_{ad},\mathcal{H}_{bc},\mathcal{H}_{bd},\mathcal{H}_{cd}$ with similar properties.* This is the statement in [31].

The circle $\mathcal{E}_d$ with the equation

$$2x^2+2y^2-(ab+ac+bc-1)x-(3a+3b+3c+abc)y+ +(a+b+c)(a+b+c+abc)=0$$

passes through the midpoint $(\frac{1}{2}a(b+c),\frac{1}{2}(2a+b+c))$ of the points $T_{AB}$ and $T_{AC}$. Because of symmetry on $a,b,c$ it is Euler's circle of the triangle $T_{AB}T_{AC}T_{BC}$, i. e. the trilateral $\mathcal{ABC}$. It intersects the $y$-axis i.e. the vertex tangent of parabola $\mathcal{P}$, in the points $Y_d=(0,a+b+c)$ and $Y_d'=(0,\frac{1}{2}(a+b+c+abc))$. The circle with the equation

$$2x^2+2y^2-(ab+ac-bc+1)x-(3a+b+c+abc)y+ +a(a+b+c+abc)=0$$

passes through the midpoint of $T_{AB}$ and $T_{AC}$, but it is incident with the midpoint $(\frac{1}{2}(ab+1),\frac{1}{2}(a+b))$ of points $S$ and $T_{AB}$ as well, so because of symmetry on $b$ and $c$ it is Euler's circle of the triangle $ST_{AB}T_{AC}$. It intersects the $y$-axis in the points $(0,a)$ and $Y_d'=(0,\frac{1}{2}(a+b+c+abc))$. Because of symmetry on $a,b,c$ it follows that the point $Y_d'$ lies on Euler's circles of the triangle $ST_{AB}T_{BC}$ and the triangle $ST_{AC}T_{BC}$, that intersects the $y$-axis residually at the point $(0,b)$ and $(0,c)$, respectively. The point $S_d'=(ab+ac+bc,a+b+c-abc)$ is symmetric to the point $S=(1,0)$ with respect to the point $S_d$ from (16). The normal from that point to the line $\mathcal{A}$ with the equation $x-ay=-a^2$ has the equation $ax+y=a^2b+a^2c+a+b+c$ and intersects the line $\mathcal{A}$ at the point $(ab+ac,a+b+c)$. The symmetry of the ordinate of this point on $a,b,c$ means that the line with the equation $y=a+b+c$ is Wallace's line of the point $S_d'$ diametrically opposite to the focus $S$ on the circle $\mathcal{S}_d$, with respect to the triangle $T_{AB}T_{AC}T_{BC}$, i. e. the trilateral $\mathcal{ABC}$. That line passes through the point $Y_d=(0,a+b+c)$ and parallel to the median of the quadrilateral $\mathcal{ABCD}$. To summarize: *The Wallace's line of the point $S_d'$ diametrically opposite to the focus S on the circle $\mathcal{S}_d$, passes through an intersection point of the Euler's circle $\mathcal{E}_d$ of trilateral $\mathcal{ABC}$ and the vertex tangent of parabola $\mathcal{P}$. It is parallel to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$.* Analogous statements are valid for Euler's circles of trilaterals $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$. Here we proved the statements taken from [**?**].

Let us the equations of $\mathcal{S}_d$ and $\mathcal{S}_c$ add after multiplying them by parameters $u$ and $v$ where $u+v=1$. We get the equation of the circle

$$x^2+y^2-[ab+(a+b)t+1]x-[a+b+(1-ab)t]y+ +ab+(a+b)t=0,$$

where $t=uc+vd$. It is easy to see that this circle passes through the points $(at,a+t)$ and $(bt,b+t)$ that are reached as the linear combinations $uT_{AC}+vT_{AD}$ and $uT_{BC}+vT_{BD}$. Hence, the statement from [20] is valid: *every circle through the focus S and the vertex $T_{AB}$ intersects lines $\mathcal{A}$ and $\mathcal{B}$ at the points that divide the segments $T_{AC}T_{AD}$, $T_{BC}T_{BD}$ in the same ratios.*

The line connecting $S_d$ from (16) with the point $T_{AB}$ from (4) has a slope $\frac{c-a-b-abc}{ac+bc-ab+1}$ and it is parallel to the line $\mathcal{D}$ that has slope $\frac{1}{d}$ under the condition $ab+cd-(a+b)(c+d)=1+abcd$, then because of symmetry on pairs of parameters $a,b$ and $c,d$ the following statements follow: *if $S_dT_{AB}$ is parallel to $\mathcal{D}$ then $S_cT_{AB}$ is parallel to $\mathcal{C}$, $S_aT_{CD}$ is parallel to $\mathcal{A}$, and $S_bT_{CD}$ is parallel to $\mathcal{B}$.* This statement is in [22]. Analogously, the following statement is valid:

**Theorem 3** *If $S_dT_{AB}$ is perpendicular to $\mathcal{D}$, the statements that $S_cT_{AB}$ is perpendicular to $\mathcal{C}$, $S_aT_{CD}$ is perpendicular to $\mathcal{A}$, and $S_bT_{CD}$ is perpendicular to $\mathcal{B}$ follow. The statement is valid for the other two possibilities of pairs on $a,b,c,d$.*

*The circle with the equation*

$$x^2+y^2-\frac{1}{2}(3+ab+ac+ad+bc+bd+cd-abcd)x-$$
$$-\frac{1}{2}(a+b+c+d-abc-abd-acd-bcd)y+$$
$$+\frac{1}{2}(1+ab+ac+ad+bc+bd+cd-abcd)=0$$

*passes through the point $S_d$ from (16), and because of symmetry on $a,b,c,d$ it passes through $S_b,S_c,S_d$.* For the first time, this statement is found in [29]. That circle is usually called Miquel's circle, but here we will call it the *central circle* of the quadrilateral $\mathcal{ABCD}$. Its equation can be written as

$$\mathcal{M}\ldots x^2+y^2-\frac{1}{2}(3+q-p)x+\frac{1}{2}(r-s)y-\frac{1}{2}(1+q-p)=0. \tag{19}$$

Obviously, it follows that it is incident with the focus $S$ from (18). Its center is the point

$$M=(\frac{1}{4}(3+q-p),\frac{1}{4}(s-r)) \tag{20}$$

that we will call a *central point* of the $\mathcal{ABCD}$, and its radius $\rho$ is given by formula $16\rho^2=(1-q+p)^2+(s-r)^2$. However, because $(a^2+1)(b^2+1)(c^2+1)(d^2+1)=(1-q+p)^2+(s-r)^2$ is valid, there is following formula

$$16\rho^2=(a^2+1)(b^2+1)(c^2+1)(d^2+1). \tag{21}$$

The line $x+dx=d^2$ is symmetric line to the line $\mathcal{D}$ with respect to the axis $\mathcal{X}$ of parabola $\mathcal{P}$. The line parallel to this line and passing through $S_d$ from (16) has the equation $x+dy=\frac{1}{2}(1+q-p)$ and intersects the axis $\mathcal{X}$ in the point $(\frac{1}{2}(1+q-p),0)$ which is because of symmetry on $a,b,c,d$ incident with the lines that pass through $S_a,S_b,S_c$ and that are parallel to the lines symmetric to $\mathcal{A},\mathcal{B},\mathcal{C}$ with respect to the axis $\mathcal{X}$. *This point is incident with the central circle $\mathcal{M}$ from (19).* This result is attributed to R. Bouvaist in [40].

Bisectors of the segments $T_{AC}T_{BC}$ and $T_{AD}T_{BD}$ have the equations

$$cx+y=\frac{1}{2}(ac+bc)c+\frac{1}{2}(a+b+2c),$$
$$dx+y=\frac{1}{2}(ad+bd)d+\frac{1}{2}(a+b+2d)$$

and the intersection point

$$T'_{AB}=\Big(\frac{1}{2}(a+b)(c+d)+1,\frac{1}{2}(a+b)(1-cd)\Big),$$

that is incident with the circle (19). The bisectors of the segments $T'_{AC}T'_{AD}$ and $T'_{BC}T'_{BD}$ are intersected in the point

$$T'_{CD}=\Big(\frac{1}{2}(a+b)(c+d)+1,\frac{1}{2}(c+d)(1-ab)\Big),$$

on the same circle. The line $T'_{AB}T'_{CD}$ with the equation $x=\frac{1}{2}(a+b)(c+d)+1$ is parallel to the directrix $\mathcal{H}$. Similarly, we get two more lines $T'_{AC}T'_{BD}$ and $T'_{AD}T'_{BC}$ parallel to $\mathcal{H}$. A line parallel to the median through the point $T'_{AB}$ has the equation $y=\frac{1}{2}(a+b)(1-cd)$, and a connecting line of the point $T_{AB}$ with the focus $S$ has the equation $(a+b)x+(1-ab)y=a+b$, and the intersection point of these lines is the point

$$T''_{AB}=\Big(\frac{1}{2}(1+ab+cd-abcd),\frac{1}{2}(a+b)(1-cd)\Big).$$

The midpoint of the points $T''_{AB}$ and $T'_{CD}$ is the central point $M$ of the quadrilateral, so *the point $T''_{AB}$ is diametrically opposite to the point $T'_{CD}$ on the central circle $\mathcal{M}$. Similarly, there are five more diameters $T''_{AC}T'_{BD},T''_{AD}T'_{BC}$, $T''_{BC}T'_{AD},T''_{BD}T'_{AC},T''_{CD}T'_{AB}$ of the circle $\mathcal{M}$.* These results are found in [31].

The line parallel to the line $\mathcal{D}$ through the point $T_{AB}=(ab,a+b)$ has the equation $x-dy=ab-ad-bd$, a connecting line of the points $S=(1,0)$ and $T_{CD}=(cd,c+d)$ has the equation $(c+d)x+(1-cd)y=c+d$, and an intersection point of these two lines is the point with coordinates

$$x=\frac{1}{\delta}(cd-1)(ad+bd-ab)+cd+d^2,$$
$$y=\frac{1}{\delta}(c+d)(1-ab+ad+bd).$$

It is easy to check that this point is incident to the circle $\mathcal{S}_d$ with equation (15). Similarly, it is valid for two more points on the circle $\mathcal{S}_d$ so the statement, [13], that *parallels to $\mathcal{D}$ through the vertices of the trilateral $\mathcal{ABC}$ intersect a circumscribed circle of the trilateral at the points, whose connecting lines to opposite vertices of the quadrilateral $\mathcal{ABCD}$ are incident with the focus of this quadrilateral* holds. Similarly, it is valid for all other trilaterals of the quadrilateral.

If two lines $\mathcal{L}$ and $\mathcal{L}'$ have slopes $\frac{m}{n}$ and $\frac{m'}{n'}$, then for the oriented angle $\angle(\mathcal{L},\mathcal{L}')$ the following formula is valid

$$\tan\angle(\mathcal{L},\mathcal{L}')=\frac{m'n-mn'}{mm'+nn'}. \tag{22}$$

Lines $ST_{AB}$ and $ST_{AC}$ have slopes $\frac{a+b}{ab-1}$ and $\frac{c+d}{cd-1}$. If $k=\frac{k}{1}$ is the slope of the line $\mathcal{T}$, then according to (22) we get

$$\tan\angle(ST_{AB},\mathcal{T})=\frac{k(ab-1)-a-b}{ab-1+k(a+b)},$$
$$\tan\angle(\mathcal{T},ST_{CD})=\frac{k(cd-1)-c-d}{cd-1+k(c+d)}.$$

The line $\mathcal{T}$ is the bisector of the lines $ST_{AB}$ and $ST_{CD}$ under the condition

$$\frac{k(ab-1)-a-b}{ab-1+k(a+b)}+\frac{k(cd-1)-c-d}{cd-1+k(c+d)}=0,$$

that by simplifying is of the form

$$(r-s)k^2+2(p-q+1)k+s-r=0. \qquad (23)$$

Symmetry on $a,b,c,d$ means that the line $\mathcal{T}$ is then the bisector of lines $ST_{AC},ST_{BD}$ and $ST_{AD},ST_{BC}$ as well. If $k_1$ and $k_2$ are slopes of the bisectors $\mathcal{T}_1$ and $\mathcal{T}_2$ of mentioned three pairs of lines, we have equalities

$$k_1+k_2=2\frac{p-q+1}{s-r}, \quad k_1k_2=-1.$$

The line with the equation $y=k(x-1)$ is incident with the point $S$ and its another intersection $T_d$ with the circle $\mathcal{S}_d$ from (15) has coordinates

$$x=\frac{1}{\kappa}[k^2+(a+b+c-abc)k+ab+ac+bc],$$
$$y=\frac{1}{\kappa}[(a+b+c-abc)k^2+(ab+ac+bc-1)k],$$

where $\kappa=k^2+1$. If that line is one bisector of $\mathcal{T}_1$ and $\mathcal{T}_2$, then in previous mentioned formulas it should be taken $k=k_1$, $k=k_2$, respectively. For other intersections $T_{d,1}$ and $T_{d,2}$ of lines $\mathcal{T}_1$ and $\mathcal{T}_2$ with the circle $\mathcal{S}_d$ we get

$$\begin{aligned}\kappa_1\kappa_2(x_2-x_1) &= ({k_1}^2+1)[{k_2}^2+(a+b+c-abc)k_2+\\ &+ab+ac+bc]-({k_2}^2+1)[{k_1}^2+(a+b+c-abc)k_1+\\ &+ab+ac+bc]=(k_1-k_2)[(a+b+c-abc)k_1k_2+\\ &+(ab+ac+bc-1)(k_1+k_2)-(a+b+c-abc)]=\\ &=(k_1-k_2)[2\frac{p-q+1}{s-r}(ab+ac+bc-1)-\\ &-2(a+b+c-abc)],\end{aligned}$$

$$\begin{aligned}\kappa_1\kappa_2(y_2-y_1) &= ({k_1}^2+1)[(a+b+c-abc){k_2}^2+\\ &+(ab+ac+bc-1)k_2]-({k_2}^2+1)[(a+b+c-\\ &-abc){k_1}^2+(ab+ac+bc-1)k_1]=(k_1-k_2)[ab+\\ &+ac+bc-1)k_1k_2-(a+b+c-abc)(k_1+k_2)-\\ &-(ab+ac+bc-1)]=(k_1-k_2)[-2(ab+ac+bc-\\ &-1)-2\frac{p-q+1}{s-r}(a+b+c-abc).\end{aligned}$$

so the line $T_{d,1}T_{d,2}$ has the slope

$$\frac{-(ab+ac+bc-1)(s-r)-(a+b+c-abc)(p-q+1)}{(ab+ac+bc-1)(p-q+1)-(a+b+c-abc)(s-r)},$$

that is equal to $-d$, because of

$$\begin{aligned}&-(ab+ac+bc-1)(s-r)-(a+b+c-abc)(p-q+1)+\\ &+d(ab+ac+bc-1)(p-q+1)-\\ &-d(a+b+c-abc)(s-r)=0.\end{aligned}$$

That means that this line is perpendicular to the line $\mathcal{D}$, and because of lines $\mathcal{T}_1$ and $\mathcal{T}_2$, it is diameter of the circle $\mathcal{S}_d$. Similarly, it is valid for the intersections of $\mathcal{T}_1$ and $\mathcal{T}_2$ with circles $\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$. We proved the statement from [40] saying:

*Diameters of the circles $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c,\mathcal{S}_d$ perpendicular to the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, respectively, intersects these circles in two points each, one of points lies on one bisector, and the other one on the other bisector of pairs of lines $ST_{AB},ST_{CD}$; $ST_{AC},ST_{BD}$ and $ST_{AD},ST_{BC}$.*

Lines $\mathcal{T}_1$ and $\mathcal{T}_2$ are so- called *Steiner's axes* of the quadrilateral $\mathcal{ABCD}$. Because of (23) they have equation

$$y=\frac{1}{r-s}[-(p-q+1)\pm\sqrt{(p-q+1)^2+(r-s)^2}](x-1).$$

The line $SM$ has the slope $\frac{r-s}{p-q+1}$, and bisectors of pairs of lines $ST_{AB},ST_{CD}$; $ST_{AC},ST_{BD}$ and $ST_{AD},ST_{BC}$ have the slope $k$ under the condition (23). For the tangent of an angle of that bisector to the line $SM$, and according to (22), we get $\frac{r-s-k(p-q+1)}{k(r-s)+p-q+1}$, that is equal to $k$ because of (23). That is the tangent of an angle of the axis $\mathcal{X}$ to this bisector. *It means that the line SM and the axis of parabola $\mathcal{P}$ are symmetric with respect to the mentioned bisector*, and that is statement from [13].

Bisectors of the sides $\mathcal{A}$ and $\mathcal{B}$ have the equations

$$\frac{1}{\sqrt{\alpha}}(x-ay+a^2)\pm\frac{1}{\sqrt{\beta}}(x-by+b^2)=0,$$

and pairs of these two lines has the equation $\beta(x-ay+a^2)^2-\alpha(x-by+b^2)^2=0$. We find the abscissae of intersections of this degenerated conics with the median $y=\frac{1}{2}s$. If we put $y=\frac{1}{2}s$ in the previous equation then coefficients next to $x^2$ and $x$ are $\beta-\alpha=-(a^2-b^2)$ and $\beta(2a^2-as)-\alpha(2b^2-bs)=2(a^2-b^2)+(ab-1)s(a-b)$, respectively, so for solutions of this equation we have equality $(x_1+x_2)=\frac{1}{a+b}(2a+2b-s-abs)$. *Because of that the midpoint $P_{AB}$ of these two intersections has the form $(\frac{1}{2(a+b)}(a+b-c-d+abs),\frac{1}{2}s)$ and it is easy to check that is collinear to the points $S=(1,0)$ and $T_{AB}=(ab,a+b)$. Similarly, it is valid for five more lines analogous to the line $T_{AB}P_{AB}$ through the focus S.* This statement can be found in [25]. That pair of bisectors of $\mathcal{A}$ and $\mathcal{B}$ intersects the axis of inscribed parabola with equation $y=0$ in the points whose abscissae are solutions of the equation $\beta(x+a^2)^2-\alpha(x+b^2)^2=0$, for them we get $x_1+x_2=2$, so these points are symmetric with respect to the focus $S=(1,0)$. That is result of [25] as well. In that paper it is proved that *the focus and point at infinity of the median are isogonal with respect to each of four trilaterals of the quadrilateral $\mathcal{ABCD}$.* For proof of this statement it is enough to prove that for example the line $ST_{AB}$ and parallel to the median $\mathcal{N}$ through the point $T_{AB}$ are lines isogonal with respect to $\mathcal{A}$ and $\mathcal{B}$, i. e. the angle of lines $\mathcal{N}$ and $\mathcal{A}$

is equal to the angle of lines $\mathcal{B}$ and $ST_{AB}$. It easily follows from the fact that these lines have slopes equal to $0 = \frac{0}{1}$ and $\frac{1}{a}$, and $\frac{1}{b}$ and $\frac{a+b}{ab-1}$, so both of these angles have the tangent angle equal to $\frac{1}{a}$ due to (22). However, [7] has already had this statement.

The point $M_d = (\frac{1}{2}(ab+ac+bc+1), \frac{1}{2}d(1-ab-ac-bc))$ is incident with the central circle (19) and with the line having equation $dx+y-d=0$, that passes through the focus $S=(1,0)$ and it is perpendicular to the line $\mathcal{D}$. Because points $M_d$ and $S_d$ from (15) have the same abscissa, then *the line $M_dS_d$ is parallel to the directrix $\mathcal{H}$ of the quadrilateral $\mathcal{ABCD}$, the same is valid for analogous lines $M_aS_a$, $M_bS_b$, $M_cS_c$.* This is result in [1].

The perpendiculars from the points $T_{AB}$ and $T_{AC}$ to the lines $\mathcal{B}$ and $\mathcal{C}$ has equations $bx+y=ab^2+a+b$, $cx+y=ac^2+a+c$, and the intersection point is the point $(ab+ac+1, a-abc)$. That point is incident with the circle $S'_a$ with equation

$$x^2+y^2-(ab+ac+ad-abcd+2)x-$$
$$-(a-abc-abd-acd)y+ab+ac+ad-abcd+1=0.$$

The intersection points of perpendiculars from points $T_{AB}$ and $T_{AD}$ to the lines $\mathcal{B}$ and $\mathcal{D}$ as well as from points $T_{AC}$ and $T_{AD}$ to lines $\mathcal{C}$ and $\mathcal{D}$ are incident with $\mathcal{S}'_a$. The circle $S'_a$ obviously is incident to the focus $S=(1,0)$. The center of that circle is the point

$$S'_a = \left(\frac{1}{2}(ab+ac+ad-abcd+2), \frac{1}{2}(a-abc-abd-acd)\right).$$

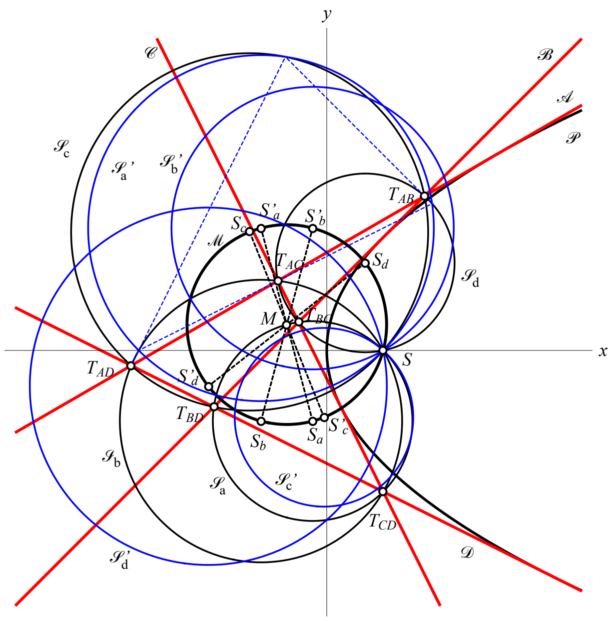


Figure 4: *Points $S'_a, S'_b, S'_c, S'_d$ are incident to the central circle $\mathcal{M}$*

The midpoint of $S'_a$ and $S_a$ from formula analogous to the formula (15) is the point $M$ from (20), the center of the central circle, so the point $S'_a$ together with the point $S_a$ is incident with that circle. Analogously, there are three more points on the central circle. The statement that *there are circles $S'_a, S'_b, S'_c, S'_d$ incident to the central circle* can be found in [43] and it is attributed to A. Hatzipolakis. Hereby, we have found out (see Figure 4) :

**Theorem 4** *The line segments $S_aS'_a, S_bS'_b, S_cS'_c, S_dS'_d$ are diameters of the central circle $\mathcal{M}$.*

Let us study now the quadrilateral $\mathcal{ABCD}'$ where $\mathcal{D}'$ is the reciprocal line to the line $\mathcal{D}$ with respect to the trilateral $\mathcal{ABC}$. The intersection point of the line $\mathcal{D}'$ and the line $\mathcal{A}$ is the point $(ab+ac-ad, a+b+c-d)$, and for the lines $\mathcal{D}'$ and $\mathcal{B}$ is the point $(ab+bc-bd, a+b+c-d)$. Two of these points and the point $T_{AB}=(ab, a+b)$ are incident to the circle

$$x^2+y^2-(2ab+ac+bc-ad-bd)x-(2a+2b+c-d-$$
$$-abc+abd)y+a^2b^2+(a+b)(a+b+c-d)=0$$

with the center $(\frac{1}{2}(2ab+ac+bc-ad-bd), \frac{1}{2}(2a+2b+c-d-abc+abd))$. This center is incident to the circle with equation

$$2x^2+2y^2-(3ab+3ac+3bc-ad-bd-cd+1+$$
$$+abcd)x-(3a+3b+3c-d-3abdc+abd+acd+$$
$$+bcd)y+a^2b^2c^2+a^2b^2+a^2c^2+b^2c^2+abcd)+$$
$$+a^2+b^2+c^2+3ad+3bd+3cd-ad-bd-cd=0,$$

and a center

$$S''_d = \Big(\frac{1}{4}(3ab+3ac+3bc-ad-bd-cd+1+abcd),$$
$$\frac{1}{4}(3a+3b+3c-d-3abc+abd+acd+bcd)\Big).$$

Out of symmetry on $a,b,c,d$ it follows that this circle is the central circle of the quadrilateral $\mathcal{ABCD}'$, so it passes through the point $S_d$. However, the midpoint of $S''_d$ and the point $M$ from (20) is the point $S_d$ from (16). Because of that the central circles of quadrilaterals $\mathcal{ABCD}$ and $\mathcal{ABCD}'$ tangent each other in the point $S_d$ and they are congruent. Similarly, it is valid for the following pairs of quadrilaterals $\mathcal{ABCD}$ and $\mathcal{ABC}'\mathcal{D}$; $\mathcal{ABCD}$ and $\mathcal{AB}'\mathcal{CD}$; and $\mathcal{ABCD}$ and $\mathcal{A}'\mathcal{BCD}$. Hence, *all five quadrilaterals $\mathcal{ABCD}$, $\mathcal{A}'\mathcal{BCD}$, $\mathcal{AB}'\mathcal{CD}$, $\mathcal{ABC}'\mathcal{D}$, $\mathcal{ABCD}'$ have the congruent central circles and the central circle of the quadrilateral $\mathcal{ABCD}$ tangents other four circles in circumcenters of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$.* These statements come from [2]. *If this circles have the radius $\rho$, then there is the circle of the radius $3\rho$ that is concentric to the central circle of the quadrilateral $\mathcal{ABCD}$, that other*

*four circles touch inside.* The statement is found in [41].
The line $\mathcal{A}$ with the equation (3) intersects the directrix $\mathcal{H}$ with the equation $x=-1$ at the point $A'=(-1,\frac{1}{a}(a^2-1))$. The connecting line of this point to the focus $S=(1,0)$ has the equation $(a^2-1)x+2ay=a^2-1$. *The altitude from the vertex $T_{BC}$ in the trilateral $\mathcal{ABC}$ has the equation $ax+y=b+c+abc$, the intersection point of these two lines is the point*

$$N_{D,A}=\Big(\frac{1}{\alpha}(1-a^2+2ab+2ac+2a^2bc),\\ \frac{1}{\alpha}(b-a+a^3+abc-a^2b-a^2c-a^3bc)\Big)$$

*for which we can check that it is incident to circumscribed circle $\mathcal{S}_d$ of this trilateral with equation (15). Analogously, the line $SA'$ intersects the altitudes to the side $\mathcal{A}$ in trilaterals $\mathcal{ABD}$ and $\mathcal{ACD}$ in the points $N_{C,A}$ and $N_{B,A}$, that are incident with the circumscribed circle of these trilaterals, respectively. Analogously, it is valid and for the lines $SB'$, $SC'$, $SD'$, where $B'$, $C'$, $D'$ are intersection points of the directrix $\mathcal{H}$ with the sides $\mathcal{B},\mathcal{C},\mathcal{D}$.* These statements are from [10].
The pedal $F$ of the normal from the focus $S$ to the directrix of the quadrilateral $\mathcal{ABCD}$ has coordinates $(-1,0)$, and point $S'$ that is diametrically opposite to the focus $S=(1,0)$ with respect to the central circle $\mathcal{M}$ has coordinates

$$S'=\left(\frac{1+q-p}{2},\frac{s-r}{2}\right).$$

Lines $FT_{AB}$ and $S'T_{CD}$ have equations

$$(a+b)x-(ab+1)y+a+b=0,\\ (c+d-a-b+r)x+(1+q-p-2cd)y=\\ =cd(c+d-a-b+r)+(c+d)(1+q-p-2cd),$$

respectively, and they are intersected in the point with coordinates

$$x=\\ \frac{(a+b)\left[c^2+d^2+c^2d^2+2cd+2p+cdp-1+ab(c^2+d^2-1)\right]}{(c+d)(a^2b^2+a^2+b^2+4ab+1)}-\\ -\frac{(c+d)(a^2+b^2-a^2b^2-1)}{(c+d)(a^2b^2+a^2+b^2+4ab+1)},\\ y=\\ \frac{(a+b)\left[(a+b)(c^2d^2+c^2+d^2+2cd-1)+2(c+d)(ab+1)\right]}{(c+d)(a^2b^2+a^2+b^2+4ab+1)}.$$

It can be checked that these coordinates fullfil the equation

$$(c+d)(x^2+y^2-1)-(c^2d^2+c^2+d^2+2cd-1)y=0$$

of the circle $SFT_{CD}$. Analogously, we can prove the rest of five statements.

Hence, the statement given in [19] is proved: *Let $F$ be the pedal of the normal from the focus $S$ to directrix of the quadrilateral $\mathcal{ABCD}$. Lines $FT_{AB}$, $FT_{AC}$, $FT_{AD}$, $FT_{BC}$, $FT_{BD}$ and $FT_{CD}$ intersect the circles $SFT_{CD}$, $SFT_{BD}$, $SFT_{BC}$, $SFT_{AD}$, $SFT_{AC}$ and $SFT_{AB}$ (except in $F$) in the points whose connecting lines with the points $T_{CD}$, $T_{BD}$, $T_{BC}$, $T_{AD}$, $T_{AC}$ and $T_{AB}$, respectively, pass through one point $S'$. This point $S'$ is diametrically opposite to the focus $S$ on the central circle.*
The line through points $T_{BC}$ from (4) and $S_d$ from (16) has the equation

$$(abc-a+b+c)x+(ab+ac-bc+1)y=\\ =ab^2c^2+ab^2+ac^2+abc+b+c.$$

Similarly, the line $T_{BD}S_c$ has the equation

$$(abd-a+b+c)x+(ab+ad-bd+1)y=\\ =ab^2d^2+ab^2+ad^2+abd+b+d \tag{24}$$

and for the intersection of these two lines we get the point with coordinates

$$S_A=(\frac{1}{\alpha}[a(abc+abd+acd-bcd+b+c+d)+1],\\ \frac{a}{\alpha}(-abcd+ab+ac+ad-bc-bd-cd+1)). \tag{25}$$

Because of symmetry of these coordinates on $b,c,d$ it follows that the line $T_{CD}S_b$ is incident to this point.
The central circle $\mathcal{M}$ with the equation (19) and circumscribed circle $\mathcal{S}_a$ of the trilateral $\mathcal{BCD}$ with equation $x^2+y^2-(bc+bd+cd+1)x-(b+c+d-bcd)y+bc+bd+cd=0$, analogous to the (15), have radical axis with the equation

$$(1+ab+ac+ad-bc-bd-cd-abcd)x+\\ +(a-b-c-d-abc-abd-acd+bcd)y+\\ +bc+bd+cd-ab-ac-ad-1+abcd=0.$$

The point $S_A$ from (25) is incident to this line, and as this point is incident to the circle $\mathcal{M}$, it is incident to the circle $\mathcal{S}_a$ as well. Similarly, it is valid for points $S_B,S_C,S_D$. Hence, *points $S_A,S_B,S_C,S_D$ are actually another intersection points of the circle $\mathcal{M}$ with circumscribed circle of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$.* A statement found in [24]: *lines $T_{BC}S_d,T_{BD}S_c,T_{CD}S_b$ are intersected in one point $S_A$ and there are three analogous points $S_B,S_C,S_D$, and these four points are incident with central circle.* On the other hand, in [30] it is proved that *these points are incident to corresponding circles $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c,\mathcal{S}_d$.* However, all these statements are found in [11] even earlier.

The line $SS_A$ has a slope

$$\frac{-ap+a(ab+ac+ad-bc-bd-cd)+a}{a(abc+abd+acd-bcd)+a(b+c+d-a)}=$$
$$-\frac{p-ab-ac-ad+bc+bd+cd-1}{abc+abd+acd-bcd+b+c+d-a}.$$

On the other hand, the connecting line of the point $M$ from (20) and $S_a$ from the formula analogous to the formula (15) has a slope

$$\frac{2(b+c+d-bcd)-(s-r)}{2(bc+bd+cd+1)-(3+q+p)}=$$
$$\frac{b+c+d-a+abc+abd+acd-bcd}{bc+bd+cd-ab-ac-ad+p-1},$$

so these two lines are perpendicular. The line $MS_a$ has the equation

$$\begin{aligned}&2(b+c+d-a+abc+abd+acd-bcd)x+\\&+2(1-abcd+ab+ac+ad-bc-bd-cd)y=\\&=bcdp+a(b^2c^2+b^2d^2+c^2d^2)+\\&+a(b+c+d)^2-2bcd-a+2(b+c+d).\end{aligned}$$

The midpoint of the point $S=(1,0)$ and the point $S_A$ from (25) has coordinates

$$\begin{aligned}x&=\frac{1}{2\alpha}[a(abc+abd+acd-bcd)+a(a+b+c+d)+2],\\y&=\frac{1}{2\alpha}[-a^2bcd+a(ab+ac+ad-bc-bd-cd)+a].\end{aligned}$$

It is easy to check that this point is incident with the line $MS_a$. Hence, *points $S$ and $S_A$ are symmetric with respect to the diameter $MS_a$ of the central circle*, and the point $S$ is incident with that circle, so because of that the point $S_A$ is incident to that circle as we have already proved it. In the same way, *pairs of points $S,S_B$; $S,S_C$; $S,S_D$ are symmetric with respect to lines $MS_b, MS_c, MS_d$*, respectively.
The perpendicular line from the point

$$\begin{aligned}S_D=\Big(&\frac{1}{\delta}[d(abd+acd+bcd-abc+a+b+c)+1],\\&\frac{d}{\delta}(-abcd+ad+bd+cd-ab-ac-bc+1)\Big)\end{aligned}$$

analogous to the point $S_A$ from (25) to the line $\mathcal{A}$ has the equation

$$\begin{aligned}&\delta(ax+y)=ad(abd+acd+bcd-abc+a+b+c)+a+\\&+d(-abcd+ad+bd+cd-ab-ac-bc+1)\end{aligned}$$

and it intersects the line $\mathcal{A}$ with the equation $\delta(x-ay)=-a^2d^2-a^2$ in the point with the coordinates

$$\begin{aligned}x=&\frac{1}{\alpha\delta}(a^3bd^2+a^3cd^2-a^3bcd+a^3d+abd^2)+\\&+\frac{1}{\alpha\delta}(acd^2-abcd+ad)\\y=&\frac{1}{\alpha\delta}(a^2bd^2+a^2cd^2-a^2bcd+a^3d^2+a^2d+ad^2)+\\&+\frac{1}{\alpha\delta}(bd^2+cd^2+a^3-bcd+a+d).\end{aligned}$$

This point is incident to the line $\mathcal{P}_D$ with the equation

$$\delta(x-dy)=(ab+ac+bc-ad-bd-cd)d^2-abcd-d^2.$$

The symmetry of this equation on $a,b,c$ means that on this line there are pedals of the perpendicular lines from the point $S_D$ to the lines $\mathcal{B}$ and $\mathcal{C}$, so $\mathcal{P}_D$ is Wallace's line of $S_D$ with respect to the trilateral $\mathcal{ABC}$. We see that this line is parallel to the line $\mathcal{D}$, as well as *Wallace's lines $\mathcal{P}_A,\mathcal{P}_B,\mathcal{P}_C$ of the points $S_A,S_B,S_C$ with respect to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ parallel to the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$*, respectively. This result is found in [15] and [14].
The line $ST_{CD}$ has the equation $(c+d)x+(1-cd)y=c+d$. It is incident with the point

$$S_{CD}=\Big(\frac{1}{2}(1+ab+cd-abcd),\frac{1}{2}(c+d)(1-ab)\Big) \qquad (26)$$

that is incident with the circle $\mathcal{M}$ from (19). Because of that $S_{CD}$ is another intersection (one is $S$) of this circle with the line $ST_{CD}$. The circle $\mathcal{S}_{CD}$ with the equation

$$\begin{aligned}&x^2+y^2-(1+ab+cd-abcd)x-(c+d)(1-ab)y+\\&+cd(1+ab-abcd)-ab(c+d)^2=0 \qquad (27)\end{aligned}$$

has the center $S_{CD}$ and the radius $\rho_{CD}$ given by ${\rho_{CD}}^2=(ab+1)^2(c^2+1)(d^2+1)$. It can be checked that this circle passes through $T_{CD}=(cd,c+d)$ and through the point $S_A$ from (25), and because of symmetry on $a$ and $b$ it is incident with $S_B$. Hence, the circle $\mathcal{S}_{CD}$ with the center $S_{CD}$ passes through the points $T_{CD},S_A,S_B$.

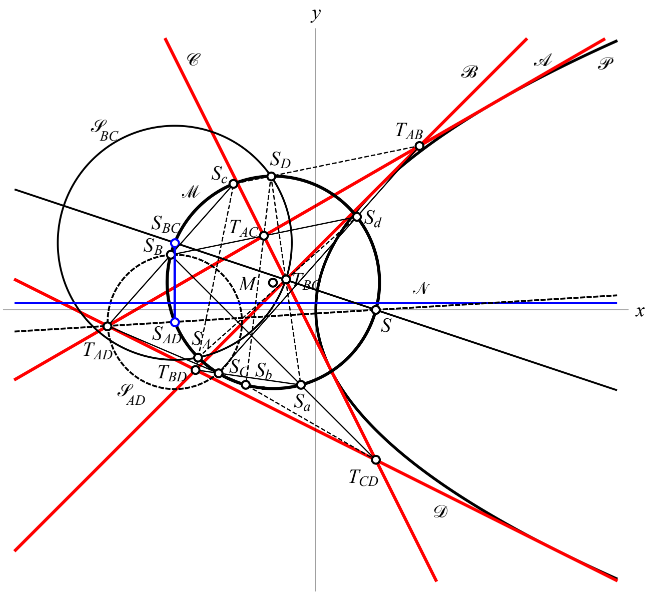


Figure 5: *An illustration of Theorem 5 on the example of the line $S_{AD}S_{BC}$*

Therefore, *if $S_{AB}, S_{AC}, S_{AD}, S_{BC}, S_{BD}, S_{CD}$ are another intersections of the circle $\mathcal{M}$ (one is $S$) with lines $ST_{AB}, ST_{AC}, ST_{AD}, ST_{BC}, ST_{BD}, ST_{CD}$, then there are circles $\mathcal{S}_{AB}, \mathcal{S}_{AC}, \mathcal{S}_{AD}, \mathcal{S}_{BC}, \mathcal{S}_{BD}, \mathcal{S}_{CD}$ with the centers $S_{AB}, S_{AC}, S_{AD}, S_{BC}, S_{BD}, S_{CD}$, that passes through the triples of points $T_{AB}, S_C, S_D$; $T_{AC}, S_B, S_D$; $T_{AD}, S_B, S_C$; $T_{BC}, S_A, S_D$; $T_{BD}, S_A, S_C$; $T_{CD}, S_A, S_B$, respectively.* The point $S_{AB}$ has the same abscissa as the point $S_{CD}$ in (26), so the line $S_{AB}S_{CD}$ has the equation $x = \frac{1}{2}(1 + ab + cd - abcd)$ and it is perpendicular to the median of the quadrilateral $\mathcal{ABCD}$. Hence, our new result is:

**Theorem 5** *If $S_{AB}, S_{AC}, S_{AD}, S_{BC}, S_{BD}, S_{CD}$ are another intersections of the circle $\mathcal{M}$ (one is $S$) with lines $ST_{AB}, ST_{AC}, ST_{AD}, ST_{BC}, ST_{BD}, ST_{CD}$ then lines $S_{AB}S_{CD}$, $S_{AC}S_{BD}$ and $S_{AD}S_{BC}$ are perpendicular to the median of the quadrilateral $\mathcal{ABCD}$.*

Let us find another intersection (except $T_{CD}$) of the line $\mathcal{C}$ and the circle $\mathcal{S}_{CD}$. Putting $x = cy - c^2$ in the equation (27), simplifying and dropping off the factor $c^2 + 1$, we get ordinate from the equation $y^2 - (2c + d - abc)y + (c + d)(c - abd) = 0$. The solution $y = c + d$ corresponds to the point $T_{CD}$, and another solution $y = c - abd$ gives $x = -abcd$, i.e. another intersection is the point $T_{CE} = (-p, c - abd)$. The circle $\mathcal{S}_{CD}$ is incident with it as well as circles $\mathcal{S}_{AC}$ and $\mathcal{S}_{BC}$ because of symmetry on $a, b, d$. This point is incident with the line $\mathcal{E}$ with the equation $x = -p$ which is perpendicular to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$. This line intersects $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ in the points $T_{AE} = (-p, a - bcd)$, $T_{BE} = (-p, b - acd)$, $T_{DE} = (-p, d - abc)$, which triplets of circles $\mathcal{S}_{AB}$, $\mathcal{S}_{AC}$, $\mathcal{S}_{AD}$; $\mathcal{S}_{AB}$, $\mathcal{S}_{BC}$, $\mathcal{S}_{BD}$; $\mathcal{S}_{AD}$, $\mathcal{S}_{BD}$, $\mathcal{S}_{CD}$, respectively, are incident with. Let us study a quadrilateral $\mathcal{ABCE}$. *The circle $\mathcal{S}_{AB}$ is incident with $T_{AB}, T_{AE}, T_{BE}$, so it is circumscribed circle to the trilateral $\mathcal{ABE}$. Similarly, the circles $\mathcal{S}_{AC}$ and $\mathcal{S}_{BC}$ are circumscribed circles to the trilaterals $\mathcal{ACE}$ and $\mathcal{BCE}$. We know from earlier that $\mathcal{S}_d$ is the circumscribed circle to $\mathcal{ABC}$. All of these four circles are incident to the point $S_D$ so it is the focus of the quadrilateral $\mathcal{ABCE}$. The centers of these circles are incident with the central circle $\mathcal{M}$, then this circle is the central circle of this quadrilateral as well. Similarly, quadrilaterals $\mathcal{ABDE}$, $\mathcal{ACDE}$, $\mathcal{BCDE}$ have focuses $S_C, S_B, S_A$, and the central circle is the circle $\mathcal{M}$ as well.* These statements are found in [36].

Hereby, we give the new result. Points $T_{AB} = (ab, a + b)$ and $T_{CE} = (-p, c - abd)$ have the midpoint $(\frac{1}{2}(ab - abcd), \frac{1}{2}(a + b + c - abd)$ that is incident with the line $\mathcal{N}_d$ with the equation $y + dx = \frac{1}{2}(a + b + c - abcd^2)$. Because of symmetry on $a, b, c$ this line is incident with midpoints of pairs of points $T_{AC}, T_{BE}$ and $T_{BC}, T_{AE}$, so $\mathcal{N}_d$ is the median of the quadrilateral $\mathcal{ABCE}$. It is perpendicular to the line $\mathcal{D}$. Similarly, it is valid for medians $\mathcal{N}_a$, $\mathcal{N}_b$, $\mathcal{N}_c$. So:

**Theorem 6** *Medians $\mathcal{N}_a$, $\mathcal{N}_b$, $\mathcal{N}_c$, $\mathcal{N}_d$ of the quadrilaterals $\mathcal{BCDE}$, $\mathcal{ACDE}$, $\mathcal{ABDE}$, $\mathcal{ABCE}$ are perpendicular to $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and intersect median $\mathcal{N}$ in one point*

$$N = \Big(-\frac{1}{2}(p+1), \frac{1}{2}s\Big). \tag{28}$$

For Theorem 6 see Figure 6.

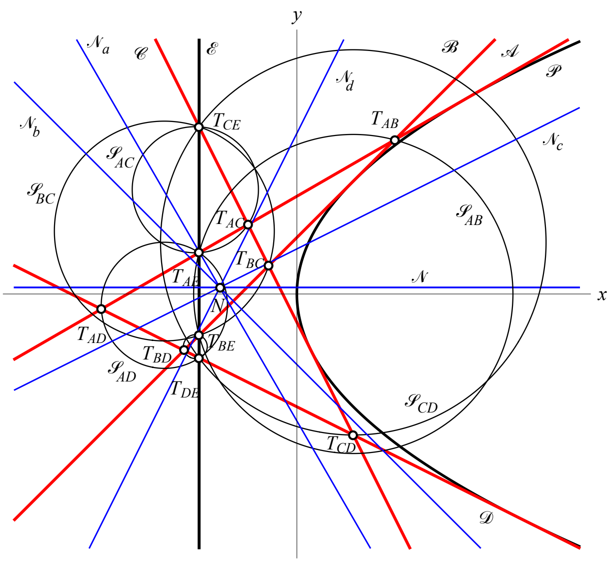


Figure 6: *Medians $\mathcal{N}_a$, $\mathcal{N}_b$, $\mathcal{N}_c$, $\mathcal{N}_d$, and $\mathcal{N}$ are intersected in the point $N$*

The line $T_{BD}S_c$ has the equation (24) and it is easy to check that is incident to $S_A$ from (25), and similarly, lines $T_{BC}S_d$ and $T_{CD}S_b$ pass through the point $S_A$. The point $T_{CE} = (-abcd, c - abd)$ and the point

$$S_{AC} = \Big(\frac{1}{2}(1 + ac + bd - abcd), \frac{1}{2}(a+c)(1-bd)\Big) \qquad (29)$$

that we achieve out of the formula (26) by a substitution $a \leftrightarrow d$, are incident with the line with the equation $(c-a)x + (1+ac)y = c - abd + ac^2 - abc^2d$, that is incident with the point $S_A$. The lines $T_{BE}S_{AB}$ and $T_{DE}S_{AD}$ are incident with $S_A$ as well. Hence, the points $S_d$, $S_c$, $S_b$, $S_{AB}$, $S_{AC}$, $S_{AD}$ are another intersections of the central circle $\mathcal{M}$ and connecting lines of the focus $S_A$ and vertices $T_{BC}, T_{BD}, T_{CD}, T_{BE}, T_{CE}, T_{DE}$ of the quadrilateral $\mathcal{BCDE}$. The point $S_d$ from (16) is analogous to the point

$$S_c = \Big(\frac{1}{2}(ab + ad + bd + 1), \frac{1}{2}(a + b + d - abd)\Big). \qquad (30)$$

It is easy to see that the connecting line $S_cS_{AC}$ have a slope equal to $\frac{1}{a}$, so that line is parallel to the line $\mathcal{A}$, i.e. it is perpendicular to the median $\mathcal{N}_a$ of the quadrilateral $\mathcal{BCDE}$. The same is valid for the lines $S_bS_{AB}$ and $S_dS_{AD}$. Analogous properties are valid for the quadrilaterals $\mathcal{ACDE}$, $\mathcal{ABDE}$ and $\mathcal{ABCE}$. The perpendicular line from the point $T_{AE} = (-p, a - bcd)$ to the line $\mathcal{B}$ has the equation $bx + y - a + bcd + ab^2cd = 0$, and a perpendicular line from the point $T_{BE}$ to the line $\mathcal{A}$ has the equation $ax + y - b + acd + a^2bcd = 0$. These two lines are intersected in the point $H_{ab} = (-abcd - cd - 1, a + b)$ which is the orthocenter of the trilateral $\mathcal{ABE}$. Hence, it is incident with directricies $\mathcal{H}_c$ and $\mathcal{H}_d$ of $\mathcal{ABDE}$ and $\mathcal{ABCE}$ that are perpendicular to the medians of these quadrilaterals and parallel to the lines $\mathcal{C}$ and $\mathcal{D}$, respectively. However, the midpoint of the point $T_{CD} = (cd, c + d)$ and the point $H_{ab}$ is the point $N$ from (28). Because of that lines $\mathcal{H}_c$ and $\mathcal{H}_d$ are symmetric to the lines $\mathcal{C}$ and $\mathcal{D}$ with respect to the point $N$. Analogously, lines $\mathcal{H}_a$ and $\mathcal{H}_b$ are symmetric to the lines $\mathcal{A}$ and $\mathcal{B}$ with respect to the point $N$. The directrix $\mathcal{H}$ of the quadrilateral $\mathcal{ABCD}$ with the equation $x = -1$ and the line $\mathcal{E}$ with the equation $x = -p$ are symmetric with respect to the point $N$. It means that the pentagonals $\mathcal{ABCDE}$ and $\mathcal{H}_a\mathcal{H}_b\mathcal{H}_c\mathcal{H}_d\mathcal{H}$ are symmetric with respect to the point $N$. The orthocenter $H_a = (-1, b + c + d + bcd)$ of the trilateral $\mathcal{BCD}$ and the intersection point $T_{AE} = (-p, a - bcd)$ of lines $\mathcal{A}$ and $\mathcal{E}$ have the midpoint $N$ from (28). Similarly, it is valid for the pairs of points $H_b, T_{BE}$; $H_c, T_{CE}$; $H_d, T_{DE}$. We have already proved that the orthocenter $H_{ab}$ of the trilateral $\mathcal{ABE}$ and the point $T_{CD}$ have the same midpoint $N$. There is a statement from [3] and [4]:

*To every quadrilateral the fifth line can be joined so that there is a point, which is common midpoint of ten segments with one endpoint in an intersection point of any two lines of these five and the another endpoint in the orthocenter of the triangle formed by the rest three lines.*

The line $\mathcal{A}$ intersects the directrix $\mathcal{H}$ in the point $A' = (-1, \frac{1}{a}(a^2 - 1))$, and the midpoint of this point and the point $T_{BC} = (bc, b + c)$ is the point $(\frac{1}{2}(bc - 1), \frac{1}{2a}(a^2 + ab + ac - 1)$. This midpoint is incident with the line $\mathcal{N}'_d$ with the equation

$$x + abcy = \frac{1}{2}[abc(a + b + c) - 1]. \qquad (31)$$

Symmetry of this equation on $a, b, c$ means that two more analogous midpoint are incident with the line $\mathcal{N}'_d$, so that line is the median of the quadrilateral $\mathcal{ABCH}$. It is incident with the point $N$ from (28), and because of symmetry on $a, b, c, d$, medians of quadrilaterals $\mathcal{ABDH}$, $\mathcal{ACDH}$, $\mathcal{BCDH}$ are incident with that point $N$ as well. This point is incident to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$. The fact that *these five medians are intersected in one point* can be found in [42]. However, we see that this point is the same point as the point $N$ from (28), so we give the new result:

**Theorem 7** *All medians of even nine quadrilaterals $\mathcal{ABCD}$, $\mathcal{ABCE}$, $\mathcal{ABDE}$, $\mathcal{ACDE}$, $\mathcal{BCDE}$, $\mathcal{ABCH}$, $\mathcal{ABDH}$, $\mathcal{ACDH}$, $\mathcal{BCDH}$ are intersected in the point $N$.*

See Figure 7.

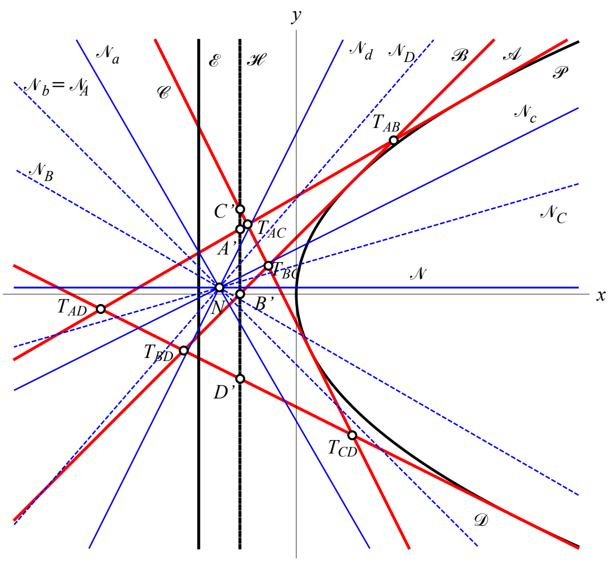


Figure 7: *Medians of quadrilaterals $\mathcal{ABCD}$, $\mathcal{ABCE}$, $\mathcal{ABDE}$, $\mathcal{ACDE}$, $\mathcal{BCDE}$, $\mathcal{ABCH}$, $\mathcal{ABDH}$, $\mathcal{ACDH}$, $\mathcal{BCDH}$ are intersected in the point $N$*

There are many more claims that are not presented in this paper and we plan to deal with them in the next paper.

## References


[1] BLANCHARD, R., Question 3755, *Mathesis* **65** (1956), 503., solutions par R. Goormaghtigh et R. Deaux, **67** (1958), 377–379.

[2] BLANCHARD, R., Note, *Mathesis* **55** (1945), 119.

[3] BRICARD, R., Question 2433, *Nouv. Ann. Math.* (4) **19** (1919), 472. without a solution

[4] BOUVAIST, R., Sur une configuration de cinq droites, *Mathesis* **56** (1947), 33–35.

[5] CLAWSON, J.W., 509. Points, lines and circles connected with the complete quadrilateral, *Math. Gaz.* **9**(129) (1917), 85–88, `https://doi.org/10.2307/3603503`

[6] CLAWSON, J.W., Problem 2898, *Amer. Math. Monthly* **28**(5) (1921), 228., `https://doi.org/10.2307/2973761`, solution by the proposer, **29**(5) (1922), 230–231, `https://doi.org/10.2307/2299751`

[7] CLAWSON, J.W., The complete quadrilateral, *Ann. of Math.* **20**(4) (1919), 232–261, `https://doi.org/10.2307/1967118`

[8] CLAWSON, J.W., Problem 2921, *Amer. Math. Monthly* **28**(10) (1921), 392., `https://doi.org/10.1080/00029890.1921.11986070`, solution by A. Pelletier, **30**(6) (1923), 339, `https://doi.org/10.2307/2300286`

[9] CONNOR, J.T., Ladies' Diary, 1795.

[10] CUNNINGHAM, A., On the circle perpend-feet pencil and orthocentrical axis of a complete quadrilateral, *Mess. Math.* **21** (1891-92), 188–191.

[11] DAVIES, T.S., Question 555, *Math. Repository* (2) **6** (1835), 229–234.

[12] DEAUX, R., Note, *Mathesis* **61** (1952), 14–16.

[13] DEAUX, R., Involution de Möbius et point de Miquel, *Mathesis* **55** (1945), 223–230.

[14] DEMIR, H., Problem 4818, *Amer. Math. Monthly* **65**(10) (1958), 779., `https://doi.org/10.1080/00029890.1958.11992001`, solution by proposer **66**(8) (1959), 732–733, `https://doi.org/10.2307/2309371`

[15] GALLATLY, W., The modern geometry of triangle, Hodgson, London 1910, 27.

[16] GAUSS, C.F., Bestimmung der grössten Ellipse, welche die vier Seiten eines gegebenen Vierecks berührt, **22** (1810), 112–121.

[17] GIBBINS, N.M., Points connected with the complete quadrilateral, *Math. Gaz.* **24**(260) (1940), 165–168, `https://doi.org/10.2307/3605707`

[18] GOORMAGHITIGH, R., Lemoyne's theorem, *Scripta Math.* **18** (1952), 182–184.

[19] GOORMAGHITIGH, R., Question 3830, *Mathesis* **67** (1958), 111., généralisation par R. Blanchard, **68** (1959), 287–288.

[20] GRINBERG, D., From the complete quadrilateral to the Droz-Farny theorem, preprint, 2005, 10.

[21] GUILLOTIN, R., Sur le cercle de Lemoyne d'une droite, *Mathesis* **64** (1955), 280–281.

[22] GUILLOTIN, R., Question 3602, *Mathesis* **62** (1953), 191., solutions M. Durieu and W. Barbenson, **63** (1954), 192–193.

[23] HEINEN, F., Auflösung der Aufgaben and Beweis der Lehrsätze 5-9 in Isten Hefte S. 96 und 29-33. im 2ten Hefte S. 191 des 2ten Baudes dieses Journals, J. Reine *Angew. Math* **3** (1828), 285–300, `https://doi.org/10.1515/crll.1828.3.285`

[24] HERMES, O., Question 476, *Nouv. Ann. Math.* **18** (1959), 171.

[25] MARCHAND, J., Géométrie du quadrilatère complet, *Bull. Soc. Vaudoise Sci. Nat.* **59** (1937), 251–270.

[26] ONO, T., Question 2258, *Nouv. Ann. Math.* (4) **15** (1915), 432., solution par R. Bouvaist, (4) **18** (1918), 114–115.

[27] OPPERMAN, A., Premiers éléments d'une théorie du quadrilatère complet, Gauthier-Villars, Paris 1919, p. 43.

[28] PITA, J.Y., Sobre la recta de los nueve puntos, *Gac. Mat.* **34** (1982), 92–98.

[29] PUISSANT, Théorème, *Corresp. Ecole Imp. Polytechn.* **1** (1806), 193.

[30] PUJET, A., É. Francoise, Solution de la question 476, *Nouv. Ann. Math.* **18** (1859), 359–362.

[31] RIPERT, L., Notes sur le quadrilatère, *Assoc. Franc. Avane. Sci.* **30** (1901), 106–118.

[32] ROCHAT, Démonstration du théorème énoncé à la paige 232 de ce volume, et de quelques autres propriétés du quadrilatère, *Ann. de Math.*, **1** (1810-11), 314–316.

[33] SCHLÖMILCH, O., Über das vollständige Viereck, *Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig*, **6** (1854), 4–13.

[34] SCOTICUS, Question 86, *Math. Repository* (2) **1** (1806), 169–170, solution by G. Pickering and J. Cavill

[35] SISKAKIS, A., Problem 1224, *Math. Mag.* **58** (1985), 238., solution by B. Poonen and H. Eves, **59** (1986), 246–247.

[36] SPORER, B., Geometrische Sätze, *Zeitschr. Math. Phys.* **31** (1886), 43–49.

[37] STEINER, J., Théorème sur le quadrilatère complet, *Ann. de Math.* **18** (1827-28), 302–304., Ges. Werke, I, 223–224.

[38] YAMOUT, J., Problem E 3299, *Amer. Math. Monthly* **95** (1988), 954., solution by H. Kappus, **98** (1991), 60.

[39] THÉBAULT, V., Sur des plans associés à un tétraèdre, *Ann. Soc. Sci. Bruxelles*, **66** (1952), 111-118; *Mathesis* **62** (1953), suppl. 8 pp.

[40] THÉBAULT, V., Sur le quadrilatère complet, *C.R. Acad. Sci. Paris* **217** (1943), 97–99.

[41] THÉBAULT, V., Question 3348, *Mathesis* **56** (1947), 243., solution par J. Andersson, **61** (1952), 62.

[42] THÉBAULT, V., Sur le quadrilatère complet, *Mathesis* **51** (1937), 187–191, 242–243.

[43] TIENHOVEN, C., Encyclopedia of Quadri-Figures, `https://chrisvantienhoven.nl/mathematics/encyclopedia`

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