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LOCI OF CENTERS IN PENCILS OF TRIANGLES IN THE ISOTROPIC PLANE

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ABSTRACT. In this paper we consider a triangle pencil in an isotropic plane consisting of those triangles that have two fixed vertices, while the third vertex is moving along a line. We study the curves of centroids, Gergonne points, symmedian points, Brocard points and Feuerbach points for such a pencil of triangles.

1. INTRODUCTION

In [7] the authors considered a triangle pencil in an isotropic plane consisting of the triangles that have the same circumcircle. They studied the loci of their centroids, Gergonne points and symmedian points, while in [6] the loci of the first and second Brocards points where observed.

In this paper we will do a similar study for the triangles that have two fixed vertices and a vertex moving along a line. Furthermore, we will extend the study with the locus of Feuerbach points.

Let us start by recalling some basic definitions and facts about the isotropic plane. It is a real projective plane where the metric is induced by a real line f and a real point F incident with it. All lines through the absolute point F are called isotropic lines, and all points incident with the absolute line f are called isotropic points. Two lines are parallel if they are incident with the same isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of points are defined by $x = \dfrac{x_1}{x_0}$, $y = \dfrac{x_2}{x_0}$, the absolute line has the equation $x_0 = 0$ and the absolute point has the coordinates $(0, 0, 1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ the distance is defined by $d(A, B) = x_B - x_A$, and for two non-parallel lines p and q, given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$, the angle is defined by $\angle(p, q) = k_q - k_p$ (see [8], [9]). The midpoint of points A and B is given by

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$\left(\frac{1}{2}(x_A + x_B), \frac{1}{2}(y_A + y_B)\right)$, while the bisector of lines p and q is given by the equation $y = \frac{1}{2}(k_p + k_q)x + \frac{1}{2}(l_p + l_q)$. A circle is defined as a conic touching the absolute line at the absolute point and it has an equation of the form $y = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$.

2. Pencil of triangles

A pencil consisting of the triangles that have two fixed vertices A, B and a vertex C moving along a line p is observed. Without loss of generality, by a suitable choice of coordinate system, we may assume that A and B have coordinates $(-1, 0)$ and $(1, 0)$. We have to distinguish between two cases: (i) p is a non-isotropic line, (ii) p is an isotropic line. In every section we shall first focus on the case (i) and assume that p is given by the equation $y = kx+l$ (with $k, l \in \mathbb{R}$), and than we will give the results for the case (ii) when p is the isotropic line with the equation $x = l$ (with $l \in \mathbb{R}$).

Let a triangle ABC with vertices

$$A = (-1, 0), \quad B = (1, 0), \quad C = (c, kc + l) \tag{2.1}$$

be given. Its sides AB, BC, CA have the equations

$$y = 0, \quad y = \frac{kc + l}{c - 1}x - \frac{kc + l}{c - 1}, \quad y = \frac{kc + l}{c + 1}x + \frac{kc + l}{c + 1}, \tag{2.2}$$

respectively. Therefore, the midpoints of its sides are

$$A_m = \left(\frac{c + 1}{2}, \frac{kc + l}{2}\right), \quad B_m = \left(\frac{c - 1}{2}, \frac{kc + l}{2}\right), \quad C_m = (0, 0), \tag{2.3}$$

and the angle bisectors b_A, b_B, b_C are given by equations

$$y = \frac{kc + l}{2(c + 1)}x + \frac{kc + l}{2(c + 1)}, \quad y = \frac{kc + l}{2(c - 1)}x - \frac{kc + l}{2(c - 1)}, \quad y = \frac{c(kc + l)}{c^2 - 1}x - \frac{kc + l}{c^2 - 1}, \tag{2.4}$$

respectively.

This pencil contains four special (singular) triangles: when C coincides with the point parallel to A or B, when C is the intersection point of p and AB, and if C is the isotropic point of the line p.

In the case (ii) when the isotropic line p is given by the equation $x = l$, $l \in \mathbb{R}$, the vertices of the triangle ABC have the coordinates

$$A = (-1, 0), \quad B = (1, 0), \quad C = (l, c), \tag{2.5}$$

and the sides have the equations

$$y = 0, \quad y = \frac{c}{l - 1}x - \frac{c}{l - 1}, \quad y = \frac{c}{l + 1}x + \frac{c}{l + 1}. \tag{2.6}$$

3. Loci of centroids and symmedian points

It follows from (2.1) and (2.3) that the three medians AA_m, BB_m and CC_m of ABC in the case (i) are given by

$$y = \frac{kc+l}{c+3}x + \frac{kc+l}{c+3}, \quad y = \frac{kc+l}{c-3}x - \frac{kc+l}{c-3}, \quad y = \frac{kc+l}{c}x, \tag{3.1}$$

respectively, and they intersect at the point

$$X_2 = \left(\frac{c}{3}, \frac{kc+l}{3}\right). \tag{3.2}$$

Instead of observing one triangle, we will observe the whole pencil of triangles. When C runs along the line p, the centroid X_2 runs along the curve k_{X2} which is already parametrized by (3.2), Figure 1. Its equation $y = kx + \frac{l}{3}$ is obtained by eliminating c from equations

$$x = \frac{c}{3}, \quad y = \frac{kc+l}{3}.$$

Therefore, the following statement holds:

Theorem 3.1. *Let the points A and B and the non-isotropic line p be given. The trace of centroid of all triangles ABC such that C lies on p is a line parallel to p.*

The symmedians are the reflections of medians in the bisectors. Therefore, we can easily calculate their equations

$$\begin{aligned} s_A \quad \ldots \quad y &= \frac{2(kc+l)}{(c+1)(c+3)}x + \frac{2(kc+l)}{(c+1)(c+3)}, \\ s_B \quad \ldots \quad y &= \frac{-2(kc+l)}{(c-1)(c-3)}x + \frac{2(kc+l)}{(c-1)(c-3)}, \\ s_C \quad \ldots \quad y &= \frac{(c^2+1)(kc+l)}{c(c^2-1)}x - \frac{2(kc+l)}{c^2-1}. \end{aligned} \tag{3.3}$$

They intersect in the symmedian point of ABC

$$S = \left(\frac{4c}{c^2+3}, \frac{2(kc+l)}{c^2+3}\right). \tag{3.4}$$

By eliminating c from the equations

$$x = \frac{4c}{c^2+3}, \quad y = \frac{2(kc+l)}{c^2+3},$$

the equation

$$(3k^2+l^2)x^2 - 12kxy + 12y^2 + 4klx - 8ly = 0 \tag{3.5}$$

of the locus k_S of symmedian points is obtained, Figure 1. Thus, we have proved:

THEOREM 3.2. *Let the points A and B and the non-isotropic line p be given. The curve of symmedian points of all triangles ABC such that C lies on p is an ellipse k_S.*

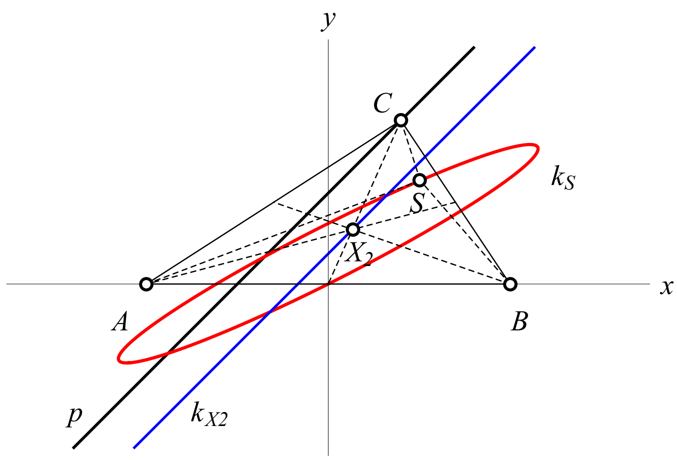

FIGURE 1. The locus k_{X2} of centroids and the locus k_S of symmedian points for the pencil of triangles with two fixed vertices A, B, and the third vertex C moving along the line p.

In the isotropic plane the center of the conic is defined as the pole of the absolute line with respect to the conic, while the axis of the conic is defined as the polar of the absolute point (see [2]). Therefore, the conic k_S has the center in the point $(0, \frac{l}{3})$ and the line with equation $y = \frac{k}{2}x + \frac{l}{3}$ is its axis. Indeed,

$$\begin{bmatrix} 0 & 2kl & -4l \\ 2kl & 3k^2+l^2 & 6k \\ -4l & -6k & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{l}{3} \end{bmatrix} = \begin{bmatrix} \frac{4l^2}{3} \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 2kl & -4l \\ 2kl & 3k^2+l^2 & 6k \\ -4l & -6k & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4l \\ -6k \\ 12 \end{bmatrix} \sim \begin{bmatrix} \frac{l}{3} \\ \frac{k}{2} \\ -1 \end{bmatrix}.$$

The center $(0, \frac{l}{3})$ of the locus k_S obviously lies on the locus k_{X2} with the equation $y = kx + \frac{l}{3}$.

The intersection points of the conic with its axis are called foci of the conic (see [9, p. 72]). Thus, the foci of k_S are points with coordinates $\left(\pm\frac{2\sqrt{3}}{3}, \frac{\pm\sqrt{3}k+l}{3}\right)$.

Let us notice that the ellipse k_S passes through the midpoint $(0,0)$ of the line segment AB. The tangent line t at that point, given by the equation $y = \frac{k}{2}x$, is parallel to the axis of the ellipse.

If the given line p passes through the midpoint $(0,0)$ of the line segment AB, i.e., $l = 0$, the locus of symmedian points k_S becomes a singular conic with the equation $(kx - 2y)^2 = 0$, the line t with multiplicity two.

We shall now observe the special case of pencil of triangles having two fixed vertices $A = (-1,0)$, $B = (1,0)$, and a vertex $C = (l,c)$ running along the isotropic line p, and prove:

THEOREM 3.3. *Let the points A and B and the isotropic line p be given. The curves of centroids and symmedian points of all triangles ABC such that C lies on p are isotropic lines.*

PROOF. Let a triangle ABC with vertices (2.5) be given. Using similar method as in the case (ii) we get the centroid and the symmedian point to be

$$X_2 = \left(\frac{l}{3}, \frac{c}{3}\right), \quad S = \left(\frac{4l}{l^2+3}, \frac{2c}{l^2+3}\right),$$

respectively. Therefore, the curves k_{X2} and k_S have equations $x = \dfrac{l}{3}$ and $x = \dfrac{4l}{l^2+3}$, respectively. □

4. Locus of Gergonne points

In order to determine the Gergonne point of the triangle ABC in case (i), we should first calculate the equation of its incircle k_i:

$$4(c^2-1)y = (kc+l)x^2 + 2c(kc+l)x + c^2(kc+l). \tag{4.1}$$

It touches the sides BC, CA, AB at the points

$$A_g = \left(c+2, \frac{(c+1)(kc+l)}{c-1}\right), B_g = \left(c-2, \frac{(c-1)(kc+l)}{c+1}\right), C_g = (-c,0), \tag{4.2}$$

respectively. The lines AA_g, BB_g, CC_g having the equations

$$\begin{aligned}
&AA_g \quad \ldots \quad y = \frac{(c+1)(kc+l)}{(c-1)(c+3)}x + \frac{(c+1)(kc+l)}{(c-1)(c+3)},\\
&BB_g \quad \ldots \quad y = \frac{(c-1)(kc+l)}{(c+1)(c-3)}x - \frac{(c-1)(kc+l)}{(c+1)(c-3)},\\
&CC_g \quad \ldots \quad y = \frac{kc+l}{2c}x + \frac{kc+l}{2},
\end{aligned} \tag{4.3}$$

respectively, intersect at one point, the Gergonne point G of the triangle ABC (see [3]) with coordinates

$$G = \left(\frac{c(c^2-5)}{c^2+3}, \frac{(c^2-1)(kc+l)}{c^2+3}\right). \tag{4.4}$$

When C moves along the line p, the Gergonne point G moves along the curve k_G whose equation

$$\begin{gathered}(3k^3+kl^2)x^3-(15k^2+l^2)x^2y+24kxy^2-12y^3\\ +(l^3-5k^2l)x^2+8ly^2-(3k^3+kl^2)x+(15k^2+l^2)y+5k^2l-l^3=0\end{gathered} \tag{4.5}$$

is obtained by eliminating c from equations

$$x=\frac{c(c^2-5)}{c^2+3}, \quad y=\frac{(c^2-1)(kc+l)}{c^2+3}.$$

This leads to:

THEOREM 4.1. *Let the points A and B and the non-isotropic line p be given. The curve of Gergonne points of all triangles ABC such that C lies on p is a rational curve of degree 3 passing through A and B.*

PROOF. Equation (4.5) is obviously the equation of a cubic. Also, it can easily be checked that the coordinates of $A=(-1,0)$ and $B=(1,0)$ satisfy that equation. A and B take the role of the Gergonne points of two special triangles in the pencil, when C is a point on p parallel to B and A, respectively. □

Figure 2 shows the incircle k_i and Gergonne point G of a triangle ABC. When the vertex C moves along the line p, the Gergonne point G moves along the cubic k_G.

The cubic k_G intersects the absolute line at a real point and a pair of conjugate imaginary points. To prove this fact, we switch to homogeneous coordinates by setting $x=\frac{x_1}{x_0}$, $y=\frac{x_2}{x_0}$. Now, (4.5) takes the form

$$\begin{gathered}(3k^3+kl^2)x_1^3-(15k^2+l^2)x_1^2x_2+24kx_1x_2^2-12x_2^3+(l^3-5k^2l)x_1^2x_0\\ +8lx_2^2x_0-(3k^3+kl^2)x_1x_0^2+(15k^2+l^2)x_2x_0^2+(5k^2l-l^3)x_0^3=0.\end{gathered}$$

After inserting $x_0=0$ we get solutions $x_2=kx_1$ and $x_2=\frac{3k\pm\sqrt{3}li}{6}x_1$, which correspond to the isotropic points $(0,1,k)$ and $(0,1,\frac{3k\pm\sqrt{3}li}{6})$. The line p touches k_G at the isotropic point $(0,1,k)$.

After doing some elementary calculations and using the tools of differential geometry, we come to the conclusion that the cubic k_G has a double point in the point

$$D=\left(\frac{9k^3-5kl^2}{l(3k^2+l^2)}, \frac{9k^4-10k^2l^2+l^4}{2l(3k^2+l^2)}\right)$$

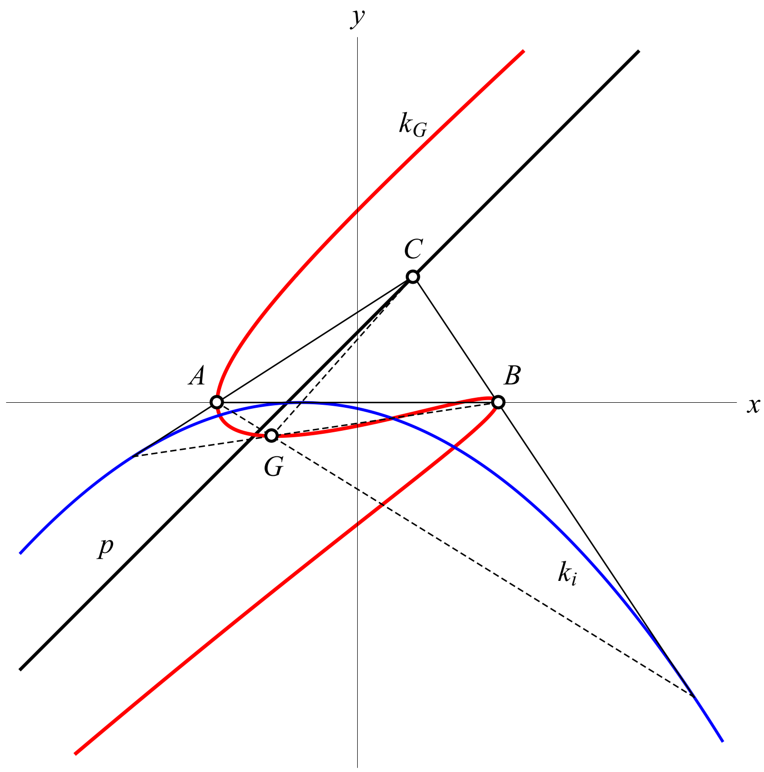

FIGURE 2. The locus k_G of Gergonne for the pencil of triangles with two fixed vertices A, B, and the third vertex C moving along the line p.

and that D is a node, an isolated double point or a cusp depending on whether $l^2 - k^2$ is greater than, less than, or equal to zero. If p passes through A ($l = k$), the cusp coincides with B, and if p passes through B ($l = -k$), the cusp coincides with A.

If the given line p passes through the midpoint $(0, 0)$ of the line segment AB, i.e., $l = 0$, two isotropic conjugate imaginary points of k_G coincide with the real isotropic point $(0, 1, \frac{k}{2})$, the isolated double point of k_G.

In the special case of pencil of triangles having two fixed vertices A, B, and a vertex C running along the isotropic line p given by the equation $x = l$, $l \in \mathbb{R}$, the following result is obtained:

THEOREM 4.2. *Let the points A and B and the isotropic line p be given. The curve of Gergonne points of all triangles ABC such that C lies on p is an isotropic line.*

PROOF. Let the triangle ABC with the vertices $A = (-1, 0)$, $B = (1, 0)$, $C = (l, c)$ be given. Using similar method as in the case (i) we obtain the coordinates of the Gergonne point

$$G = \left(\frac{l(l^2-5)}{l^2+3}, \frac{c(l^2-1)}{l^2+3} \right)$$

and the equation of the curve k_G

$$x = \frac{l(l^2-5)}{l^2+3}.$$

$\square$

5. Locus of Feuerbach points

In [1] and [10, pp. 109-115] the authors in different ways proved that in a triangle in the isotropic plane the midpoints of the sides and the feet of the altitudes lie on a circle, so-called Euler circle. They also proved that the incircle k_i and Euler circle k_e touch each other in a point which is called the Feuerbach point of the triangle ABC.

Let us now determine the equation of the Euler circle k_e for the triangle with vertices (2.1) and midpoints (2.3). It can be easily checked that k_e has the equation

$$(c^2-1)y = -2(kc+l)x^2 + 2c(kc+l)x. \tag{5.1}$$

It follows from (4.1) and (5.1) that the Feuerbach point is

$$\Phi = \left(\frac{c}{3}, \frac{4c^2(kc+l)}{9(c^2-1)} \right). \tag{5.2}$$

It can be noticed that Φ is parallel to the centroid X_2 given by (3.2). By eliminating c from the equations

$$x = \frac{c}{3}, \quad y = \frac{4c^2(kc+l)}{9(c^2-1)},$$

the equation

$$12kx^3 - 9x^2y + 4lx^2 + y = 0 \tag{5.3}$$

of the locus k_Φ of Feuerbach points is obtained.

Figure 3 shows the incircle k_i, Euler circle k_e and Feuerbach point Φ of a triangle ABC. When the vertex C moves along the line p, the Feuerbach point Φ moves along the cubic k_Φ.

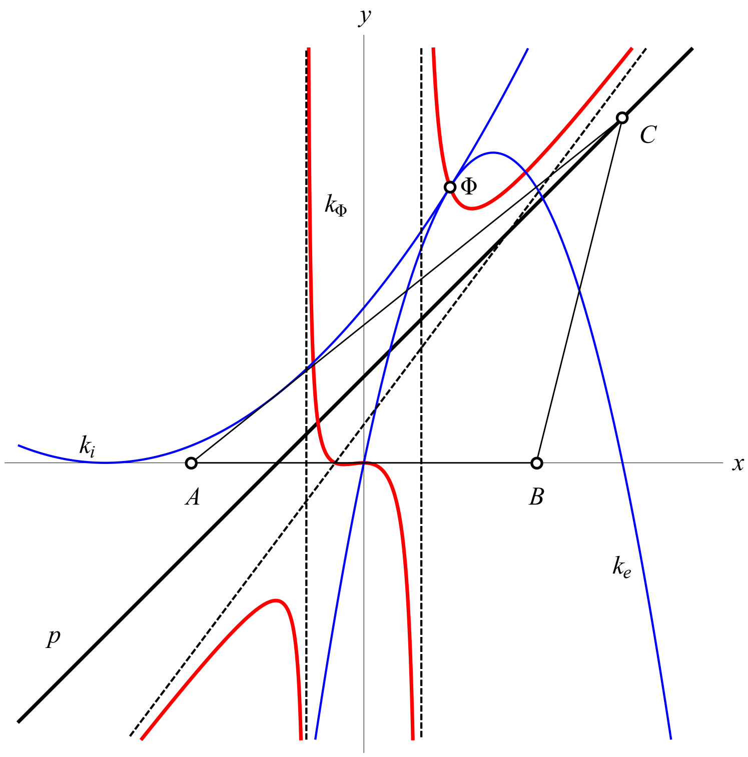

FIGURE 3. The locus k_Φ of Feuerbach points for the pencil of triangles with two fixed vertices A, B, and the third vertex C moving along the line p.

THEOREM 5.1. *Let the points A and B and the non-isotropic line p be given. The curve of the Feuerbach points of all triangles ABC such that C lies on p is 2-circular rational curve of degree 3 having an ordinary double point at the absolute point.*

PROOF. To determine the intersection points of the absolute line and k_Φ, we can write (5.3) in homogenous coordinates as

$$12kx_1^3 - 9x_1^2x_2 + 4lx_1^2x_0 + x_2x_0^2 = 0. \tag{5.4}$$

Since the absolute line is given by $x_0 = 0$ we get two solutions, a double solution $x_1 = 0$ and solution $x_2 = \frac{4}{3}kx_1$. Therefore, the absolute point $F = (0,0,1)$ is the intersection point with the intersection multiplicity 2, and $M = (0,1,\frac{4}{3}k)$ is the intersection point with the intersection multiplicity 1. The absolute point F can have intersection multiplicity 2 in two ways, either it is a double point of k_Φ or it is a regular point in which k_Φ touches the absolute line. We will prove that the first case is true and we will determine the tangents of k_Φ at F. Every line through F has the equation of the form $x = m$, or in terms of homogeneous coordinates $x_1 = mx_0$. It intersects the cubic (5.4) in the points whose coordinates (x_0, x_1, x_2) satisfy equation

$$x_0^2 \left(4m^2(3km + l)x_0 + (1 - 9m^2)x_2\right) = 0. \tag{5.5}$$

Therefore, $x_0 = 0$ is a double root and F is the intersection point counted twice. The solution $x_0 = 0$ will be a triple root and F, the intersection point, counts three times if, and only if, $m = \pm\frac{1}{3}$. We can conclude that F is a node at which k_Φ has tangents $x = \pm\frac{1}{3}$. The absolute point takes the role of the Feuerbach point of two special triangles in the pencil when C is the point parallel to A or B, i.e., $c = \mp 1$, respectively. □

Let us now determine the tangent line of k_Φ at its isotropic point $M = (0, 1, \frac{4}{3}k)$. Every line through M has equation of the form $y = \frac{4}{3}kx + n$, i.e., $x_2 = \frac{4}{3}kx_1 + nx_0$. It intersects the cubic (5.4) in the points whose coordinates satisfy equation

$$x_0 \left((4l - 9n)x_1^2 + (\frac{4}{3}kx_1 + nx_0)x_0 \right) = 0.$$

When $n = \frac{4l}{9}$, $x_0 = 0$ is a double root. Thus, the line with equation $y = \frac{4}{3}kx + \frac{4l}{9}$ is the tangent line at the isotropic point M.

The curve k_Φ intersects the fixed line AB in the points satisfying equations (5.3) and $y = 0$. From $4x^2(3kx + l) = 0$ we get intersection points $C_m = (0, 0)$ counted twice, and the point $(-\frac{l}{3k}, 0)$. When C approaches the point in which p meets AB, the Feuerbach point of the triangle ABC approaches the obtained point $(-\frac{l}{3k}, 0)$.

In the cases when p passes through A or B, the cubic k_Φ splits onto an isotropic line and a special hyperbola. To prove this fact, we will assume that p passes through A which is precisely when $l = k$. The equation (5.3) turns into

$$(3x + 1)(4kx^2 - 3xy + y) = 0.$$

The equation $3x + 1 = 0$ represents an isotropic line, while the equation $4kx^2 - 3xy + y = 0$ represents the conic which intersects the absolute line at the absolute point and the isotropic point of the line $y = \frac{4k}{3}x$. Similarly, if p passes through B, k_Φ splits onto the isotropic line with equation $3x - 1 = 0$ and the special hyperbola with the equation $4kx^2 - 3xy - y = 0$.

In the case (ii) when the vertices of the triangle ABC have coordinates given by (2.5), using similar method as above, we get the Feuerbach point to be

$$\Phi = \left(\frac{l}{3}, \frac{4cl^2}{9(l^2 - 1)} \right).$$

The locus of Feuerbach points is, therefore, given by the equation

$$x = \frac{l}{3}$$

and the following theorem holds:

THEOREM 5.2. *Let the points A and B and the isotropic line p be given. The curve of Feuerbach points of all triangles ABC such that C lies on p is an isotropic lines.*

We can see that the isotropic lines k_{X2} and k_ϕ coincide.

6. Loci of Brocard points

It was shown in [6] that for every triangle in the isotropic plane there exist the first and second Brocard point, and they are unique. The *first Brocard point* B_1 is defined as the point such that its connections with the vertices A, B, C form equal angles with the sides AB, BC, and CA, respectively. The angle h is called the *first Brocard angle*. Analogously, the *second Brocard point* B_2 is defined as the point such that its connection lines with the vertices A, B, C form equal angles with the sides AC, BA, and CB, respectively. The angle is called the *second Brocard angle* and equals $-h$. The concept of Brocard points was introduced in the isotropic plane as an analogue of the concept of Brocard points in Euclidean plane given in [4].

If the vertices of the triangle ABC are given by (2.1), the first Brocard point is

$$B_1 = \left(\frac{c^2 + 4c - 1}{c^2 + 3}, \frac{4(c+1)^2(kc+l)}{(c^2+3)^2} \right), \tag{6.1}$$

and the second Brocard point is

$$B_2 = \left(\frac{-c^2 + 4c + 1}{c^2 + 3}, \frac{4(c-1)^2(kc+l)}{(c^2+3)^2} \right). \tag{6.2}$$

Indeed, some elementary calculations show that

$$\angle(AB_1, AB) = \angle(BB_1, BC) = \angle(CB_1, CA) = \frac{-2(kc+l)}{c^2+3},$$

and

$$\angle(AB_2, AC) = \angle(BB_2, BA) = \angle(CB_2, CB) = \frac{2(kc+l)}{c^2+3}.$$

The expressions

$$x = \frac{c^2 + 4c - 1}{c^2 + 3}, \quad y = \frac{4(c+1)^2(kc+l)}{(c^2+3)^2}$$

present the parametrization of the locus of the first Brocard points, an obviously rational quartic curve k_{B1}. By eliminating c we get an implicit equation of k_{B1}:

$$\begin{aligned}(3k^2+l^2)x^4 &+ 4k(k+l)x^3 + 4(l-3k)x^2y + 2(k-l)^2x^2 \\ &- 8(k+l)xy + 16y^2 - 4k(k+l)x + 4(k-3l)y - k^2 - 4kl + l^2 = 0.\end{aligned} \tag{6.3}$$

Analogously, it can be shown that the equation of the locus of the second Brocard points is

$$\begin{aligned}(3k^2+l^2)x^4 &+ 4k(l-k)x^3 + 4(l+3k)x^2y - 2(k+l)^2x^2 \\ &+ 8(l-k)xy + 16y^2 + 4k(k-l)x - 4(k+3l)y - k^2 + 4kl + l^2 = 0.\end{aligned} \tag{6.4}$$

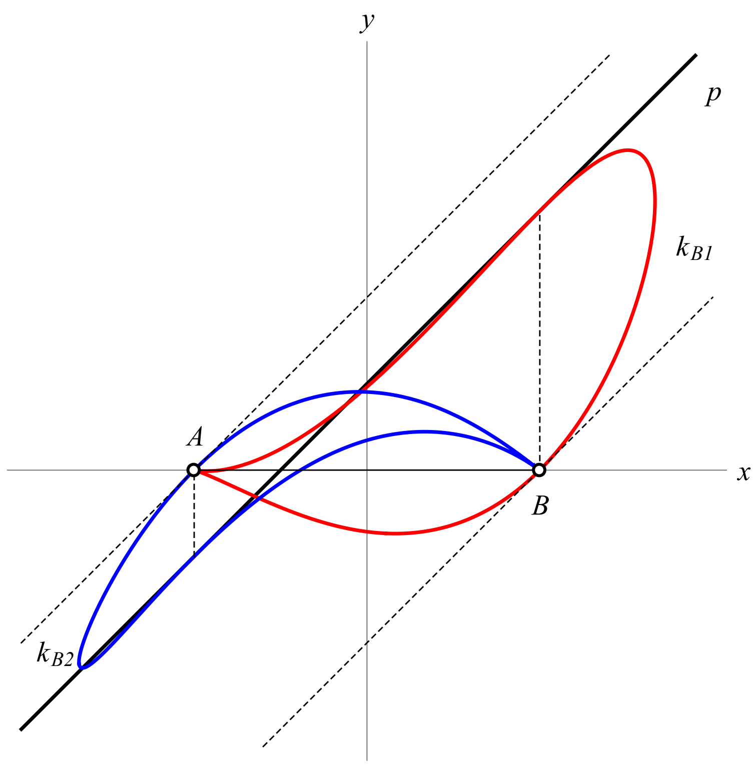

FIGURE 4. The locus k_{B1} of the first Brocard points and the locus k_{B2} of the second Brocard points for the pencil of triangles with two fixed vertices A, B, and the third vertex C moving along the line p.

THEOREM 6.1. *Let the points A and B and the non-isotropic line p be given. The curve of the first Brocard points (the second Brocard points) of all triangles ABC such that C lies on p is an entirely circular and rational curve of degree 4 with a cusp at A (B). It touches p at the point parallel to B (A), and the tangent line at B (A) is parallel to p.*

PROOF. We will prove the theorem for the curve of the first Brocard points. The curve k_{B1} given by (6.3) is obviously a curve of degree 4. Therefore, it intersects every line in 4 points. This particularly holds for the absolute line f. To study the behavior of k_{B1} at the absolute line, we need to switch to homogenous coordinates, similarly as we did in the proof of Theorem 5.1. After inserting $x_0 = 0$ (intersection with the absolute line) into the equation of k_{B1}, we get $(3k^2 + l^2)x_1^4 = 0$. Thus, $x_1 = 0$ is fourfold solution and $F = (0, 0, 1)$ is the only intersection point of k_{B1} and f. This proves that k_{B1} is an entirely circular quartic. Detailed studies of circular quartics in the isotropic plane were given in [5].

It can be easily checked that every line through A, i.e., the line with the equation of the form $y = mx + m$, intersects k_{B1} in A counted two times. Indeed, after inserting $y = mx + m$ into (6.3), we get

$$(1+x)^2\left(-k^2-4kl+l^2+4km-12lm+1m^2-2(k^2-2kl+l^2+6km-2lm)x+(3k^2+l^2)x^2\right)=0.$$

This leads to the conclusion that $x = -1$ is a double solution. It is a triple solution precisely when $m = \frac{l-k}{2}$. Thus, the tangent line at the cusp A is given by $y = \frac{l-k}{2}x + \frac{l-k}{2}$.

The line through B parallel to p has the equation $y = kx - k$. From (6.3) we get

$$(x-1)^2\left(11k^2 + 8kl + l^2 - 2(k^2 - 4kl - l^2)x + (3k^2 + l^2)x^2\right) = 0.$$

Thus, $B(1, 0)$ is the intersection point with intersection multiplicity 2. We conclude that the tangent line of k_{B1} at B is given by $y = kx - k$.

To determine the intersection points of k_{B1} and p we need to insert $y = kx + l$ into (6.3) and we get

$$(x-1)^2\left(-k^2 + 5l^2 - 2(k^2 - 4kl - l^2)x + (3k^2 + l^2)x^2\right) = 0.$$

It follows that the line p touches k_{B1} in the point $(1, k+l)$ parallel to B. This completes the proof. ∎

Let us also notice that in the four special cases when C becomes parallel to A, when C becomes parallel to B, when C becomes the intersection point of p and AB, and when C becomes an isotropic point, the first Brocard point of ABC is the point A, the point parallel to B (the contact point of k_{B1} and p), the intersection point of k_{B1} and AB, and the point B, respectively.

We will now consider the cases when the loci of Brocard points degenerate. When p passes through B, i.e., $l = -k$, the equation (6.3) turns into

$$4(kx^2 - 2y - k)^2 = 0.$$

Thus, the curve k_{B1} of the first Brocard points degenerates onto a circle with multiplicity two. The line p is the tangent line of the circle at B. Similarly, if p passes through A, the curve k_{B2} of the second Brocard points degenerates onto a circle with multiplicity two.

At the and observe the special case of pencil of triangles having two fixed vertices $A = (-1, 0)$, $B = (1, 0)$, and a vertex $C = (l, c)$ running along the isotropic line p given by the equation $x = l$, $l \in \mathbb{R}$.

Theorem 6.2. *Let the points A and B and the isotropic line p be given. The curves of Brocard points of all triangles ABC such that C lies on p are isotropic lines.*

Proof. Let a triangle ABC with vertices (2.5) be given. Using similar method as in the previous sections we get the first and the second Brocard point to be

$$B_1 = \left(\frac{l^2 + 4l - 1}{l^2 + 3}, \frac{4c(l + 1)^2}{(l^2 + 3)^2}\right), \quad B_2 = \left(\frac{-l^2 + 4l + 1}{l^2 + 3}, \frac{4c(l - 1)^2}{(l^2 + 3)^2}\right),$$

respectively. Therefore, the curves k_{B1} and k_{B2} have equations $x = \dfrac{l^2 + 4l - 1}{l^2 + 3}$ and $x = \dfrac{-l^2 + 4l + 1}{l^2 + 3}$, respectively. □

References

[1] J. Beban-Brkić, R. Kolar-Šuper, Z. Kolar-Begović and V. Volenec, *On Feuerbach's theorem and a pencil of circles in the isotropic plane*, J. Geom. Graph. **10** (2006), 125–132.

[2] J. Beban-Brkić, M. Šimić and V. Volenec, *On foci and asymptotes of conics in the isotropic plane*, Sarajevo J. Math. **3** (2007), 257–266.

[3] J. Beban-Brkić, V. Volenec, Z. Kolar-Begović and R. Kolar-Šuper, *On Gergonne point of the triangle in isotropic plane*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **17** (2013), 95–106.

[4] A. Bernhart, *Polygons of pursuit*, Scripta Math. **24** (1959), 23–50.

[5] E. Jurkin, *Circular quartics in the isotropic plane generated by projectively linked pencils of conics*, Acta Math. Hung. **130** (2011), 35–49.

[6] E. Jurkin, *Curves of Brocard points in triangle pencils in isotropic plane*, KoG **22** (2018), 20–23.

[7] M. Katić Žlepalo and E. Jurkin, *Curves of centroids, Gergonne points and symmedian centers in triangle pencils in isotropic plane*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **22** (2018), 119–127.

[8] R. Kolar-Šuper, Z. Kolar-Begović, V. Volenec and J. Beban-Brkić, *Metrical relationships in a standard triangle in an isotropic plane*, Math. Commun. **10** (2005), 149–157.

[9] H. Sachs, Ebene Isotrope Geometrie, Wieweg, Braunschweig/Wiesbaden, 1987.

[10] I. M. Yaglom, A Simple Non-Euclidean Geometry and its Physical Basis, Heidelberg Science Library, Springer-Verlag, New York - Heidelberg - Berlin, 1979.

Geometrijska mjesta središta u pramenovima trokuta u izotropnoj ravnini

Ema Jurkin

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