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Jurkin, Ema

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BISECTORS OF CONICS IN THE ISOTROPIC PLANE

Ema Jurkin

Abstract. In this paper we introduce the concept of a bisector of two curves in an isotropic plane. We study the bisectors of conics and classify them according to their type of circularity. In general the bisector of conics is a quartic, while in the case when both conics are circular, its degree decreases.

1. Introduction

An isotropic plane is a projective plane with a distinct line f and an incident distinct point $F \in f$. The line f is called the *absolute line* and F is called the *absolute point*. The lines incident with F form the pencil of *isotropic lines*. Any pair (A, B) of distinct points joined by an isotropic line, i.e., collinear with F, is said to be *parallel*.

Isotropic transformations are those projective automorphisms of the projective plane that leave the absolute figure (f, F) invariant. The standard affine model of the isotropic plane is obtained by setting $f : x_0 = 0$ and $F = 0 : 0 : 1$. In this model, isotropic lines are given by the equations $x = const.$ and pairs of parallel points (A, B) are characterized by $x_A = x_B$. The isotropic distance $d(A, B)$ of a pair of non-parallel points is defined by $d(A, B) = x_B - x_A$, and for pairs of (distinct) parallel points $(x_A = x_B, y_A \neq y_B)$, the span $s(A, B) = y_B - y_A$ serves as a substitute of the isotropic distance. Further, it is possible to define an isotropic angle between any pair of non-isotropic lines $p : y = k_p x + l_p$ and $q : y = k_q x + l_q$ as $\angle(p, q) = k_q - k_p$, cf. [4], [5].

The isotropic distance, the span, and the isotropic angle are invariant under all isotropic isometries, i.e., the 3-parametric group of affine transformations of the form $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}$, where $a, b, c \in \mathbb{R}$.

Midpoints can be defined in a natural way as $\big(\frac{1}{2}(x_A + x_B), \frac{1}{2}(y_A + y_B)\big)$, while the bisector of lines p and q is given by the equation $y = \frac{1}{2}(k_p + k_q)x +$

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$\frac{1}{2}(l_p + l_q)$. If an isotropic line i intersects lines p, q and their bisector in points A, B and T respectively, then $s(A,T) = s(T,B)$ i.e. T is the harmonic conjugate of the absolute point F with respect to A, B, see Fig. 1. This fact will be used as a definition of the bisector of the algebraic curves of general degree.

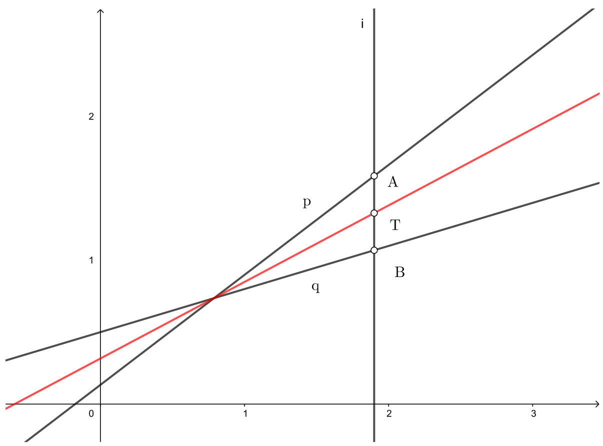

FIGURE 1. The bisector of the lines p and q in the affine model of the isotropic plane.

According to their position with respect to the absolute figure, conics are classified into ellipses, hyperbolas, special hyperbolas, parabolas and circles, as explained in [2] and [5]. An ellipse is a conic that intersects the absolute line in a pair of complex conjugate points, while a hyperbola intersects it in two different real points. If one of the two intersection points with the absolute line falls into the absolute point, the conic is called a special hyperbola. A conic touching the absolute line is called a parabola. If a conic touches the absolute line at the absolute point, it is said to be a circle.

A curve in the isotropic plane is circular if it passes through the absolute point. Its degree of circularity is defined as the number of its intersection points with the absolute line falling into the absolute point. If it does not share any further point with the absolute line except the absolute point, it is entirely circular, as defined in [3].

Ellipses, hyperbolas and parabolas are non-circular conics. Special hyperbolas are 1-circular, while circles are 2-circular conics.

A circular cubic can be 1, 2 or 3-circular. A 1-circular cubic passes through the absolute point F and meets the absolute line f in two further points (those two points can be distinct or fall together). A 2-circular cubic touches f at its regular point F or it has a double point in F at which f is not a tangent. A cubic is 3-circular if f osculates it at F or f is a tangent at a double point F.

Similarly, a circular quartic can be 1, 2, 3 or 4-circular. Moreover, there are various different kinds of 2-, 3- and 4-circularities depending on the type of singularity falling into F and, additionally, on the relative position of f with regard to the curve. The absolute line can intersect it, touch it, osculate or hyperosculate it at the absolute point. The absolute point can be a regular, a double or a triple point of the curve. Therefore, each type of singularity gives rise to various degrees and types of circularities.

In this paper we extend the concept of the bisector of two lines to the bisector of two curves. The focus is put on the bisectors of conics which we classify according to the type of circularity.

2. Bisector of curves

In this section we generalize the concept of the bisector of straight lines to the bisector of curves of higher degrees.

Let the curves $\mathcal{A}$ and $\mathcal{B}$ of degrees m and n, respectively, be given. Each isotropic line intersects $\mathcal{A}$ in m points A_i, $i = 1 \dots m$, and $\mathcal{B}$ in n points B_j, $j = 1, \dots n$. Let T_{ij} be points such that $\mathrm{H}(A_i, B_j, F, T_{ij})$. The locus $\mathcal{K}$ of all points T_{ij} for all isotropic lines is called the *bisector curve* of $\mathcal{A}$ and $\mathcal{B}$. Since each isotropic line intersects $\mathcal{K}$ in $m \cdot n$ points, we can conclude that $\mathcal{K}$ is a curve whose degree is at most $m \cdot n$. In general $\mathcal{K}$ is a curve of degree $m \cdot n$, while in the case when both initial curves $\mathcal{A}$ and $\mathcal{B}$ are circular, its degree decreases.

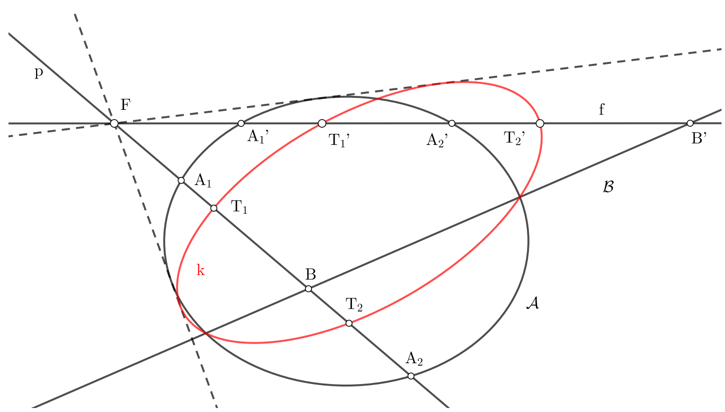

Figure 2. The bisector $\mathcal{K}$ of the hyperbola $\mathcal{A}$ and the line $\mathcal{B}$ from the projective point of view.

In Fig. 2 the bisector $\mathcal{K}$ of the conic $\mathcal{A}$ and the non-isotropic line $\mathcal{B}$ is depicted. The absolute line f intersects $\mathcal{B}$ in the point B', and $\mathcal{A}$ in the points A_1' and A_2'. Depending on the type of the conic $\mathcal{A}$, the points A_1' and A_2' can be complex conjugate, real, fall together, or one of them or both coincide with the absolute point F. Therefore, the points T_1' and T_2' are of the same type. Thus, we conclude:

THEOREM 2.1. *Let a conic $\mathcal{A}$ and a non-isotropic line $\mathcal{B}$ be given. The bisector of $\mathcal{A}$ and $\mathcal{B}$ is a conic of the same type as $\mathcal{A}$.*

PROOF. Additionally to the synthetic proof given above, we offer an analytic proof for the affine model of the isotropic plane.

Let the conic $\mathcal{A}$ be given by the equation of the form

$$\mathcal{A} \ldots a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0.$$

If $\mathcal{A}$ is a non-circular conic ($a_{22} \neq 0$), its equation can be rewritten as

$$\mathcal{A} \ldots y^2 + 2p_1(x)y + p_2(x) = 0,$$

where p_i, q_i are polynomials of degree i in x, i.e., $p_1(x) = a_{12}x + a_{02}$, $p_2(x) = a_{11}x^2 + 2a_{01}x + a_{00}$. Let the line $\mathcal{B}$ be given by the equation of the form

$$y = kx + l.$$

After intersecting $\mathcal{A}$ and $\mathcal{B}$ with the isotropic line

$$x = t,$$

we get the intersection points

$$A_{1,2} = \left(t, -p_1 \pm \sqrt{p_1^2 - p_2}\right), \quad B = (t, kt + l)\,,$$

respectively. Therefore, the points T_i, such that $\mathrm{H}(A_i, B, F, T_i)$, $i = 1, 2$, have coordinates

$$\left(t, \frac{kt + l - p_1 \pm \sqrt{p_1^2 - p_2}}{2}\right).$$

The implicitization of any of the two above parametrizations yields one conic $\mathcal{K}$ with the equation

$$(2y - kx - l + p_1)^2 = p_1^2 - p_2,$$

where $p_1(x) = a_{12}x + a_{02}$, $p_2(x) = a_{11}x^2 + 2a_{01}x + a_{00}$. $\mathcal{K}$ is a hyperbola, a parabola or an ellipse depending on whether $a_{12} - a_{11}$ is greater than, equal to, or less than zero. Thus, $\mathcal{K}$ is of the same type as $\mathcal{A}$.

If $\mathcal{A}$ is a circular conic ($a_{22} = 0$), its equation can be rewritten as

$$\mathcal{A} \ldots 2p_1(x)y + p_2(x) = 0.$$

After intersecting $\mathcal{A}$ and $\mathcal{B}$ with the isotropic line $x = t$, we get the intersection points

$$A = \left(t, -\frac{p_2}{2p_1}\right), \quad B = (t, kt + l),$$

respectively. The point T, such that $\mathrm{H}(A, B, F, T)$, has coordinates

$$\left(t, \frac{2(kt + l) - p_2}{4p_1}\right).$$

The implicitization of the above parametrization yields the conic $\mathcal{K}$ with the equation

$$2p_1(2y - kx - l) + p_2 = 0,$$

where $p_1(x) = a_{12}x + a_{02}$, $p_2(x) = a_{11}x^2 + 2a_{01}x + a_{00}$. Homogenizing ($x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$) the equation of $\mathcal{K}$ and setting $x_0 = 0$ yields the intersection with the absolute line f as

$$(a_{11} - 2ka_{12})x_1^2 + 4a_{12}x_1x_2 = 0.$$

Thus, F is the intersection point with the intersection multiplicity 1. F is the intersection point of f and $\mathcal{K}$ with the intersection multiplicity 2 if and only if $a_{12} = 0$, which is precisely in the case when $\mathcal{A}$ is a circle. □

We should also notice that $\mathcal{K}$ passes through the intersection points of $\mathcal{A}$ and $\mathcal{B}$, and that $\mathcal{A}$ and $\mathcal{K}$ share the isotropic tangents.

3. Bisectors of pairs of conics

In the focus of our study are bisectors of two conics. We classify them according to the type of their intersections with the absolute line.

Let $\mathcal{A}$ and $\mathcal{B}$ be two 0-circular conics. If at least one of them is an ellipse, then due to the definition of the bisector curves, all four isotropic points T'_{ij}, $i, j \in \{1, 2\}$, are complex points. If both $\mathcal{A}$ and $\mathcal{B}$ are hyperbolas, then they intersect f in real points A'_1, A'_2, B'_1 and B'_2, respectively, and T'_{ij}, $i, j \in \{1, 2\}$, are also real points. Thus, $\mathcal{K}$ is a 0-circular quartic with four real intersections with the absolute line. If $\mathcal{A}$ is a hyperbola and $\mathcal{B}$ parabola, then A'_1, A'_2 are two different real points, and B'_1 and B'_2 coincide, see Fig. 3. Therefore, the points T'_{11} and T'_{12} fall into one point T'_1, and the points T'_{21} and T'_{22} into the other point T'_2. Thus, $\mathcal{K}$ touches f in two different points. In the case when both $\mathcal{A}$ and $\mathcal{B}$ are parabolas, we have $A'_1 = A'_2$ and $B'_1 = B'_2$, and $\mathcal{K}$ has only one intersection point with the absolute line f, see Fig. 4. It is a tacnode with f as a tangent, classed among singularities of a higher order, as explained in [6]. This observation can be summarized to:

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two 0-circular conics. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is a 0-circular quartic. Its isotropic points are of the following types:*

- *If at least one of conics $\mathcal{A}$ and $\mathcal{B}$ is an ellipse, $\mathcal{K}$ intersects f in complex conjugate points.*

- *If both $\mathcal{A}$ and $\mathcal{B}$ are hyperbolas, $\mathcal{K}$ intersects f in four real points.*
- *If $\mathcal{A}$ is a hyperbola and $\mathcal{B}$ is a parabola, $\mathcal{K}$ touches f at two points.*
- *If both $\mathcal{A}$ and $\mathcal{B}$ are parabolas, $\mathcal{K}$ has a double point on f at which both tangents coincide with f.*

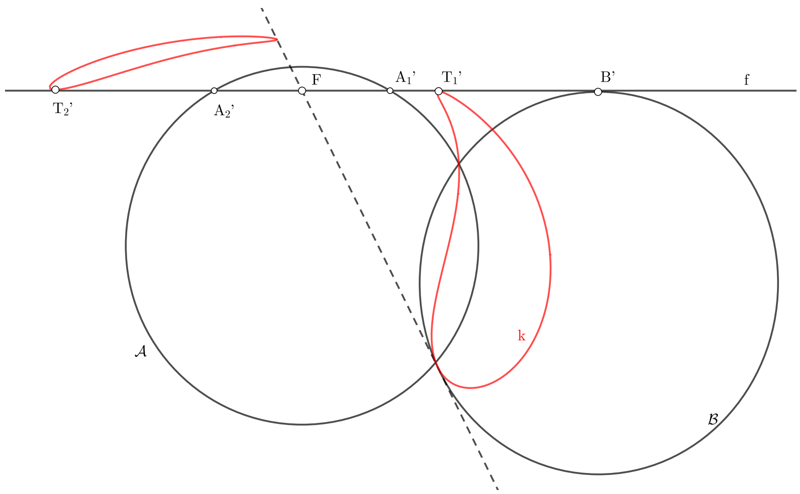

FIGURE 3. The bisector $\mathcal{K}$ of the hyperbola $\mathcal{A}$ and the parabola $\mathcal{B}$ from the projective point of view.

Let us now determine the equation of the bisector $\mathcal{K}$ of the pair of non-circular conics in the affine model of the isotropic plane. Let conics $\mathcal{A}$ and $\mathcal{B}$ be given by the equations of the form

$$\mathcal{A} \ldots a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0,$$
$$\mathcal{B} \ldots b_{00} + b_{11}x^2 + b_{22}y^2 + 2b_{01}x + 2b_{02}y + 2b_{12}xy = 0.$$

For non-circular conics $\mathcal{A}$ and $\mathcal{B}$ $(a_{22}, b_{22} \neq 0)$ these equations can be rewritten as

$$\mathcal{A} \ldots y^2 + 2p_1(x)y + p_2(x) = 0, \tag{3.1}$$

$$\mathcal{B} \ldots y^2 + 2q_1(x)y + q_2(x) = 0, \tag{3.2}$$

where p_i, q_i are polynomials of degree i in x, i.e.

$$p_1(x) = a_{12}x + a_{02}, \quad p_2(x) = a_{11}x^2 + 2a_{01}x + a_{00},$$
$$q_1(x) = b_{12}x + b_{02}, \quad q_2(x) = b_{11}x^2 + 2b_{01}x + b_{00}. \tag{3.3}$$

After intersecting $\mathcal{A}$ and $\mathcal{B}$ with the isotropic line

$$x = t, \tag{3.4}$$

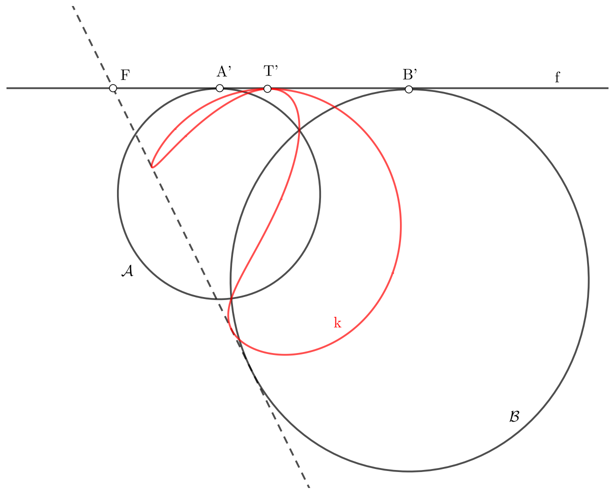

FIGURE 4. The bisector $\mathcal{K}$ of the parabolas $\mathcal{A}$ and $\mathcal{B}$ from the projective point of view.

we get the intersection points

$$\left(t, -p_1 \pm \sqrt{p_1^2 - p_2}\right), \quad \left(t, -q_1 \pm \sqrt{q_1^2 - q_2}\right),$$

respectively. The points T_{ij}, $i, j \in \{1, 2\}$, have coordinates

$$\left(t, \frac{-p_1 \pm \sqrt{p_1^2 - p_2} - q_1 \pm \sqrt{q_1^2 - q_2}}{2}\right),$$

for four combinations of signs.

The implicitization of any of the four above parametrizations yields one quartic curve with the equation

$$\left[(2y + p_1 + q_1)^2 - p_1^2 - q_1^2 + p_2 + q_2\right]^2 = 4(p_1^2 - p_2)(q_1^2 - q_2), \tag{3.5}$$

where p_i, q_i are given by (3.3).

Let $\mathcal{A}$ be a 1-circular and let $\mathcal{B}$ be a 0-circular conic. Each isotropic line p intersects $\mathcal{B}$ at two points, and $\mathcal{A}$ in the absolute point F and a further point. Thus, two intersection points of p and $\mathcal{K}$ fall into F. This means that F is the intersection point of $\mathcal{K}$ and each line passing through F with the intersection multiplicity 2. F is therefore a double point of $\mathcal{K}$. The tangent t of $\mathcal{A}$ at F is at the same time the tangent of $\mathcal{K}$ at F. Depending on whether $\mathcal{B}$ intersects t at two real points, a pair of complex conjugate points, or it touches it, $\mathcal{K}$ has a node, an isolated double point or a cusp at F, see Fig. 5. Beside the absolute point F, the absolute line f intersects $\mathcal{K}$ at two points. They are different and real, a pair of complex conjugate points, or they fall together depending on whether $\mathcal{B}$ is a hyperbola, an ellipse or a parabola. Now we can state the following theorem:

THEOREM 3.2. *Let $\mathcal{A}$ be a 1-circular and $\mathcal{B}$ a 0-circular conic. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is 2-circular quartic having a double point at F. Its isotropic points are of the following types:*

- *If $\mathcal{B}$ is an ellipse, $\mathcal{K}$ intersects f in a pair of complex conjugate points.*
- *If $\mathcal{B}$ is hyperbola, $\mathcal{K}$ intersects f in two real points.*
- *If $\mathcal{B}$ is a parabola, $\mathcal{K}$ touches f.*

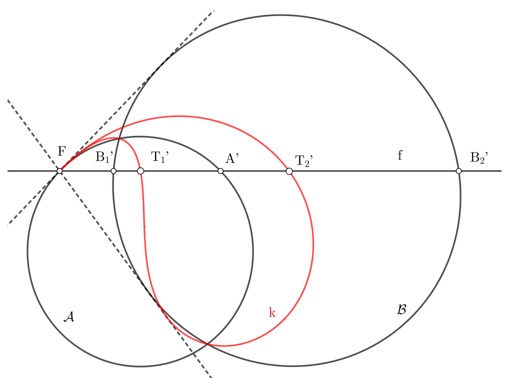

FIGURE 5. The bisector $\mathcal{K}$ of the special hyperbola $\mathcal{A}$ and the hyperbola $\mathcal{B}$ with a common isotropic tangent from the projective point of view.

If $\mathcal{A}$ is a circular conic, $a_{22} = 0$, then its equation takes the form

$$\mathcal{A} \ldots 2p_1(x)y + p_2(x) = 0. \tag{3.6}$$

The isotropic line $x = t$ intersects the circular conic $\mathcal{A}$ given by (3.6) and the non-circular conic $\mathcal{B}$ given by (3.2) in the points

$$\left(t, -\frac{p_2}{2p_1}\right), \quad \left(t, -q_1 \pm \sqrt{q_1^2 - q_2}\right),$$

respectively, and points T_1, T_2 have coordinates

$$\left(t, \frac{1}{2}\left(-q_1 \pm \sqrt{q_1^2 - q_2} - \frac{p_2}{2p_1}\right)\right).$$

After implicitization of the above parametrizations, the following equation of one quartic curve is obtained

$$[2p_1(2y - q_1) + p_2]^2 = 4p_1^2(q_1^2 - q_2). \tag{3.7}$$

Let $\mathcal{A}$ be a circle, and let $\mathcal{B}$ be a 0-circular conic. In Fig. 6 the bisector of the circle $\mathcal{A}$ and the hyperbola $\mathcal{B}$ is shown. Every isotropic line intersects $\mathcal{B}$ in two points B_1 and B_2, and $\mathcal{A}$ in the absolute point F and further point A. Due to the definition of the bisector of curves, two points of $\mathcal{K}$ fall into F, and the other two in points T_i such that $\mathrm{H}(A, B_i, F, T_i)$, $i = 1, 2$. So, F is the intersection point of multiplicity two of $\mathcal{K}$ and each line through F. F is therefore a double point of $\mathcal{K}$. Since the absolute line f has no other intersection points with $\mathcal{A}$ except the absolute point F, the same holds for $\mathcal{K}$. Thus, $\mathcal{K}$ is an entirely circular quartic. Depending on whether $\mathcal{B}$ is a hyperbola, a parabola or an ellipse, F is a node, a cusp or an isolated double point. We can now state:

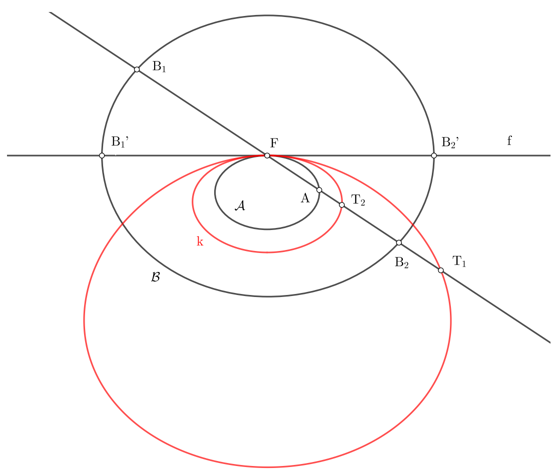

FIGURE 6. The bisector $\mathcal{K}$ of the circle $\mathcal{A}$ and the hyperbola $\mathcal{B}$ from the projective point of view.

THEOREM 3.3. *Let $\mathcal{A}$ be a circle and $\mathcal{B}$ a 0-circular conic. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is an entirely circular quartic having a double point at the absolute point at which both tangents coincide with the absolute line.*

If $\mathcal{A}$ is a circle, i.e., $a_{22} = a_{12} = 0$, $a_{02} \neq 0$, then its equation can be rewritten as

$$\mathcal{A} \ldots y + p_2(x) = 0, \tag{3.8}$$

and the following equation of the bisector $\mathcal{K}$ of the circle $\mathcal{A}$ and the non-circular conic $\mathcal{B}$ given by (3.2) is obtained

$$(2y - q_1 + p_2)^2 = q_1^2 - q_2. \tag{3.9}$$

In Fig. 7 (left) the bisector $\mathcal{K}$ of the circle $\mathcal{A}$ with the equation $y = x^2$ and the hyperbola $\mathcal{B}$ with the equation $x^2 - y^2 = 1$ is depicted. It is an entirely circular quartic with the equation $x^4 - 4x^2y - x^2 + 4y^2 + 1 = 0$ having a double point at the absolute point F. The absolute line f touches both branches of $\mathcal{K}$.

The bisector of the circle $\mathcal{A}$ with the equation $y = x^2$ and the ellipse $\mathcal{B}$ with the equation $x^2 + y^2 = 4$ is an entirely circular quartic given by the equation $x^4 - 4x^2y + x^2 + 4y^2 - 4 = 0$. It has an isolated double point in the absolute point F, see Fig. 7 (right).

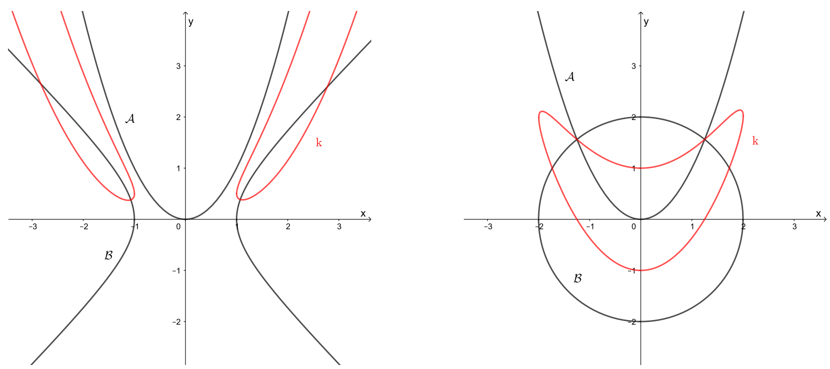

FIGURE 7. The bisector $\mathcal{K}$ of the circle $\mathcal{A}$ and the hyperbola/ellipse $\mathcal{B}$.

In the end, we should consider the cases when both $\mathcal{A}$ and $\mathcal{B}$ are circular, i.e., $a_{22} = b_{22} = 0$. The equations (3.1) and (3.2) take the forms

$$\mathcal{A} \ldots 2p_1(x)y + p_2(x) = 0, \tag{3.10}$$

$$\mathcal{B} \ldots 2q_1(x)y + q_2(x) = 0. \tag{3.11}$$

The intersection points A, B of the isotropic line $x = t$ and conics $\mathcal{A}$, $\mathcal{B}$ are the points

$$\left(t, -\frac{p_2}{2p_1}\right), \quad \left(t, -\frac{q_2}{2q_1}\right),$$

respectively, and point T, such that $\mathrm{H}(A, B, F, T)$ holds, has coordinates

$$\left(t, -\frac{p_2 q_1 + p_1 q_2}{4 p_1 q_1}\right).$$

The implicitization of the above parametrization yields a cubic curve with the equation

$$4 y p_1 q_1 + p_2 q_1 + p_1 q_2 = 0. \tag{3.12}$$

After inserting (3.3) in (3.12) we get

$$\begin{aligned} 4y(a_{12}x + a_{02})(b_{12}x + b_{02}) + (a_{11}x^2 + 2a_{01}x + a_{00})(b_{12}x + b_{02}) + & \\ +(a_{12}x + a_{02})(b_{11}x^2 + 2b_{01}x + b_{00}) &= 0. \end{aligned} \tag{3.13}$$

Homogenizing ($x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$) the latter equation and setting $x_0 = 0$ yields the intersection with the absolute line as

$$4 a_{12} b_{12} x_1^2 x_2 + (a_{11} b_{12} + a_{12} b_{11}) x_1^3 = 0. \tag{3.14}$$

It can be concluded that the absolute point F is the intersection of $\mathcal{K}$ and the absolute line with the intersection multiplicity 2, and the other intersection point is the ideal point of the line $y = -\frac{a_{11}b_{12} + a_{12}b_{11}}{4a_{12}b_{12}} x$. Thus, $\mathcal{K}$ is a 2-circular cubic. It can easily be shown that every line through F intersects $\mathcal{K}$ in the absolute point with the intersection multiplicity 2, i.e. F is a double point of $\mathcal{K}$. Therefore, the following theorem holds:

THEOREM 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two special hyperbolas. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is a 2-circular cubic having a double point at the absolute point.*

If $\mathcal{A}$ is a circle given by (3.8) and $\mathcal{B}$ is a circular conic given by (3.11), then (3.12) takes the form

$$4 q_1 y + 2 p_2 q_1 + q_2 = 0 \tag{3.15}$$

and (3.14) turns into $a_{11} b_{12} x_1^3 = 0$. Thus, F is the only intersection point of $\mathcal{K}$ and f. We conclude:

THEOREM 3.5. *Let $\mathcal{A}$ be a circle and let $\mathcal{B}$ be a special hyperbola. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is an entirely circular cubic having a double point at the absolute point.*

In Fig. 8 the bisector $\mathcal{K}$ of the circle $y = x^2$ and special hyperbola $xy = 1$ is depicted. It is an entirely circular cubic $x^3 - 2xy + 1 = 0$ with a double point at F. The absolute line f and the line $x = 0$ are the tangents of $\mathcal{K}$ at F.

If both conics $\mathcal{A}$ and $\mathcal{B}$ are circles, i.e., $a_{12} = b_{12} = 0$, their equations can be rewritten as

$$y + p_2(x) = 0, \quad y + q_2(x) = 0,$$

respectively, where p_2 and q_2 are polynomials given by (3.3). Now (3.12) takes the form

$$2y + p_2(x) + q_2(x) = 0,$$

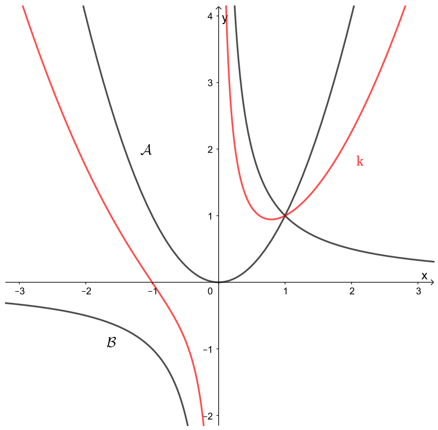

FIGURE 8. The bisector $\mathcal{K}$ of the circle $\mathcal{A}$ and the special hyperbola $\mathcal{B}$.

i.e.,

$$2y + (a_{11} + b_{11})x^2 + 2(a_{01} + b_{01})x + a_{00} + b_{00} = 0,$$

that obviously represents a circle. Therefore, we proved:

THEOREM 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two circles. The bisector $\mathcal{K}$ of $\mathcal{A}$ and $\mathcal{B}$ is a circle.*

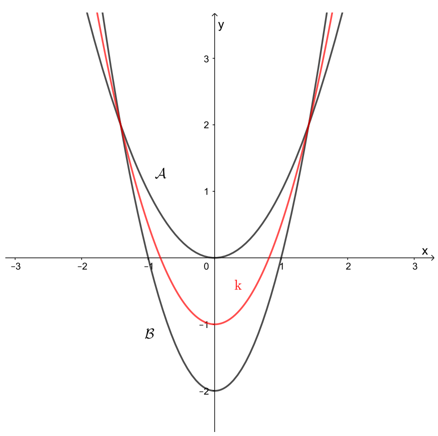

FIGURE 9. The bisector $\mathcal{K}$ of the circles $\mathcal{A}$ and $\mathcal{B}$.

The bisector $2y = 3x^2 - 2$ of the circles $y = x^2$ and $y = 2x^2 - 2$ is shown in Fig. 9.

4. Future work

In this paper we have extended the definition of the bisector of two straight lines to the bisector of two curves in the isotropic plane. In the Euclidean plane the bisector of two lines is a pair of straight lines each of which encloses the same angle with either line, but at the same time, it is the set of points equidistant from the two given lines, as explained in [1]. This fact is not valid in the isotropic plane. Our definition of bisectors is based on the span, not on the distance. Therefore, it should be also interesting to study the set of points equidistant from two curves.

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Simetrale konika u izotropnoj ravnini

Ema Jurkin

Sažetak. U radu uvodimo pojam simetrale dviju krivulja u izotropnoj ravnini. Proučavamo simetrale konika i klasificiramo ih prema tipu cirkularnosti. Simetrala dviju konika je u općem slučaju krivulja četvrtog reda. Ako su obje konike cirkularne, red simetrale se smanjuje.

Ema Jurkin
Faculty of Mining, Geology and Petroleum Engineering
University of Zagreb
10 000 Zagreb, Croatia
E-mail: ema.jurkin@rgn.unizg.hr

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